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INVENTORY MODELS WITH MULTIPLE ORDER OPPORTUNITIES

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INVENTORY MODELS WITH MULTIPLE ORDER OPPORTUNITIES

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To Lin, George and Roger

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ABSTRACT

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We examine the structure of the optimal ordering policies for several inventory models with multiple order opportunities. The inventory models that are examined in this thesis are: 1) the newsvendor model with a second order opportunity; 2) the stochastic multi-period inventory model with limited order opportunities; and 3) the stochastic multi-period inventory model with two supply modes.

For 1), we develop three models (Models I, II, and III) that differ in the timing of the second order. In all three models, the first order is placed for delivery at the beginning of the season. In Model I, the second order quantity is determined at the beginning of the season for delivery at a pre-specified time. In Model II, the second order quantity is determined at some pre-specified time that can be any where during the season. In Model III, both the timing and quantity of the second order are determined dynamically.

For Model III, we establish for the first time that under appropriate conditions, the decision to place a second order is characterized by a time-dependent (s,S) policy. A counterexample is presented that suggests that the policy structure under more general conditions would likely be more complex. Model II is a generalization of the model of Fisher et al. (2001). We show that that under mild regularity conditions, this problem has sufficient structure to

reduce to a sequential search of two programs, each of which has at most one local minimum. For Model I, we establish robust conditions under which the optimization behaves well.

For 2), we examine a model under the assumptions that shortages are backordered, demand density functions fall in PF_2 family and unit purchase cost is non-decreasing as we get closer to the end of the problem horizon. We show that a time varying (s,S) policy is the optimal decision rule for this model.

For 3), the two supply modes differ in their delivery leadtimes. The chapter contributes by showing that there are unique "order up to" levels to determine the order quantities from these two suppliers. We identify conditions when it is optimal to order from just one supplier or from both. In case it is optimal to order from both in a period, we show that at the beginning of the period, if the beginning inventory level is between a certain pair of points, then it is optimal to raise the inventory position to the higher point through a slow order. However, if the beginning inventory position is lower than the lower point, then the inventory level is first raised up to this point through a fast order and then the inventory position is raised up to the higher point through a slow order. If the beginning inventory is higher than the higher point, no order needs to be placed. The optimal policies in this chapter are supported by the property that the cost is unimodal in the beginning inventory position and convex in the beginning inventory level. We need the PF_2 density assumption to prove this property.

CHAPTER 1. INTRODUCTION AND THESIS OVERVIEW

1.1. Introduction

The overall goal of this research is to examine the structure of optimal ordering policies for several inventory models with multiple order opportunities. The inventory models that are examined in this thesis are: 1) the newsvendor model with a second order opportunity; 2) the stochastic multi-period model with limited order opportunities; and 3) the stochastic multi-period model with two supply modes.

The traditional newsvendor model assumes a single order opportunity before the start of a selling season facing a random demand so that the expected cost of meeting the demand by the order is minimized. We develop a discrete-time model of the selling season. We study the impact of having an additional order opportunity on the structure of the optimal policy in terms of the first order quantity, time to place the second order and the second order quantity.

The stochastic multi-period inventory model refers to a sequence of decisions – how much to order – at the beginnings of the periods each facing a random demand, which is reveled after the order decision, so that the expected cost of meeting demand in the periods is minimized. We study the impact of the "scarcity of order opportunities" on the structure of the optimal policy in terms of time to place an order and how much if ordering. By scarcity of order opportunities, we mean that the number of orders that can be placed is less than or equal to the number of periods.

The stochastic multi-period model with two supply modes refers to a sequence of decisions – how much to order from each supply mode – at the beginning of every period facing a random demand, so that the total expected cost over the problem horizon is minimized. We study the structure of the optimal policy, in terms of two order quantities, when one supply mode has instant delivery and the other one has less-than-1-period delivery leadtime.

Demand is independent across periods. An important class of demand density frequently used in this thesis is the PF_2 density. This class of density includes Exponential distribution, Gamma with $\alpha \geq 1$, Weibull with $\alpha \geq 1$, Truncated normal, Normal, Laplace, Exponential family, Noncentral F-distribution, Noncentral t-distribution, Noncentral chi-square, Kolmogoroff-Smiroff distribution, Uniform on $(0,1)$ etc. Thus PF_2 includes most of realistic demand density functions.

1.2. Thesis Overview

To study the newsvendor problem with a second order opportunity, we have developed three models that differ in the timing of the second order. In all of the three models, the first order is placed for delivery at the beginning of the season. In Model I, the second order quantity is determined at the beginning of the season for delivery at a specified time. In Model II, the second order quantity is determined at some pre-specified time that can be any where during the season. In Model III, both the timing and quantity of the second order are determined dynamically.

In the most general setting that we consider (Model III), an initial order is received at the start of the selling season; subsequently, a second order with a positive delivery leadtime is placed if conditions so warrant. Until the second order is received, as much as possible of demand is filled from inventory-on-hand or from pipeline inventory; additional demand is treated as lost sales. For this fully dynamic setting, in Theorem 4, we establish for the first time that under appropriate conditions, the decision to place a second order is characterized by a time-dependent two-number (s,S) policy: at each decision epoch during the season, a second order is placed to bring the inventory position to the optimal target S, only if the inventory position is below the threshold s. A counterexample is presented that suggests that the policy structure under more general conditions would likely be more complex.

The analysis of this fully dynamic model exploits the optimal policy structure of the partially dynamic Model II in which the order quantity but not order timing is determined dynamically. This model is a generalization of the model of Fisher et al. (2001) since unlike them we allow backorder costs to depend on the duration of the fulfillment delay. While Fisher et al. demonstrated that this cost minimization problem is not convex; we show that that under mild regularity conditions (Theorem 3), this problem has sufficient structure to reduce the optimization problem to a sequential search for two programs, each of which has at most one local minimum.

Finally, Model II and Model III build on the structural properties of the fully static Model I in which both orders are placed before the start of the season. This model is closely related to the models of Donohue (2000), Iyer et al. (2003), and Barankin (1961). Theorem 1 identifies the elegant structure of this optimization problem which is a pre-requisite to establishing conditions under which the underlying optimization problem is well behaved. Hence, Theorem 1 leads to Theorem 2 that establishes robust conditions under which the optimization behaves well.

Since our representation of the problem allows for both backorders and lost sales, altogether these three model variations provides an integrated view of the growing literature on the newsvendor model with a second order opportunity.

To study the stochastic multi-period problem with limited order opportunities, we consider a discrete-time inventory decision setting where the number of orders that can be placed is less than the number of periods that can be used for order placement. Facing this constraint, the decision is how to optimally place orders in terms of ordering time and quantity. Recall that the number of periods is greater than the number of orders; the decision maker needs to optimally allocate the limited order opportunities. A key feature of this decision is optimally and dynamically allocating these opportunities. We show that a time varying (s,S) policy is the optimal decision rule under the assumptions that shortages are backordered, demand density functions fall in PF_2 family and unit purchase cost is non-decreasing in the ordering time. Under this rule, the decision maker places an order to raise the inventory level up to S whenever it is less than s and there is an order opportunity left. In fact, s and S depend on the time and the number of order opportunities left.

It is worth mentioning that in Model III of Chapter 2, the decision on the second order shares the feature of the model in Chapter 3 that limited order opportunities are to be optimally utilized. However they differ in the way the shortages are treated. Recall in Model III of Chapter 2, after the placement of the second order, as many as possible of demand unmet from on-hand inventory are accepted as backorders as far as they can be satisfied from the second order delivery, and any additional demand are lost sales. In the model analyzed in Chapter 3, all shortages are accepted as backorders. This difference leads to the different result that the structure of the optimal policy for Model III may be complex when the delivery leadtime for the second order is positive while the structure of the optimal policy for the model in Chapter 3 is a simple (s,S) policy under suitable conditions.

To study the stochastic multi-period inventory problem with two supply modes, we consider a discrete-time inventory decision setting where two orders are placed at the beginning of each period: one order has an instant delivery (referred as "fast order" hereafter) and the other one has a less-than-a-period delivery time (referred as "slow order" hereafter). Under such a setting, the general question is how to make optimal inventory decisions in each period, in terms of the inventory level and the inventory position (or, equivalently, the quantities for the fast order and the slow order). Under some conditions, we show that if the beginning inventory level is between a certain pair of points, then it is optimal to raise the inventory position to the higher point through a slow order; However if the beginning inventory level is lower than the lower point, then the inventory level is first raised up to this point through a fast order and then the inventory position is raised up to the higher point through a slow order. If the beginning inventory is higher than the higher point, no order needs to be placed.

CHAPTER 2. NEWSVENDOR PROBLEM WITH A SECOND ORDER OPPORTUNITY

2.1. Introduction

Traditionally, in many industrial settings ranging from agriculture to fashion, the decision of how much to procure of a product is made well before the start of a well-defined selling season. Consequently, the decision-maker is unable to take advantage of subsequent information that becomes available as the season draws closer. As exemplified by its pioneering use in the 1980's by Benetton, the fashion retail giant, providing a second order opportunity around the start of the season, can significantly reduce markdowns and leftover inventory. Propelled by this and related innovations in supply chain management, researchers have begun a re-examination of analytical models of inventory management that support such decision-making. Much of this effort has focused on incorporating a second order opportunity into the single-period newsvendor model that determines the optimal order quantity under demand uncertainty. The range of applications has been extensive. It includes models on quick response (Fisher and Raman 1996), catalog sales (Eppen and Iyer 1997), planning hybrid seed inventories (Jones et al. 2001) and electric power (Iyer et al. 2003).

2.1.1. Overview of Events and Decision Variables

Our goal in this chapter is to provide a comprehensive treatment of the newsvendor problem with a second order opportunity. At the start of the pre-season, as in the conventional newsvendor problem, the first order is placed for delivery at the start of the selling season. Until the second order is placed, as much as possible of the demand is filled from inventory-on-hand; the rest is backordered (incurring shortage costs), with a commitment to fill it from the second order. Using the updated demand distribution and the current status of its stock, the decision maker places the second order for delivery after a pre-specified leadtime in the season. The retailer continues to accept demand until all inventory on-hand and on-order is depleted. Demand that cannot be filled from the second order, is considered lost, or effectively, satisfied from outside the system. Hence, under this framework the decision maker must make three inter-related choices: the first order quantity, when to place the second order, and the second order quantity.

2.1.2. Three Models

This broad view on the timing of the procurement decisions leads to the three models that are presented and analyzed in this study. Together these three models capture this rich problem domain that allows for updating demand distributions between the two ordering opportunities, and allows the choice between backordering demand and lost sales. In Model I, the static case is considered: all three decisions are made before the start of the season without the benefit of any updated information. In Model II, the partially dynamic case is considered: the first order quantity is chosen, as in Model I, for delivery at the start of the season. Subsequently, the second order if there is one is placed at a pre-announced time, allowing the decision maker to make a more refined decision since it takes into account the status of the current stock and a more refined understanding of demand uncertainty. If the second order is placed at the start of the season, only information from the preseason is used to determine the revised demand distribution; if the second order is placed during the season, the revised distribution is updated from information collected during the preseason and the observed demand in the initial selling season. Finally, Model III captures the fully dynamic case since both the timing and quantity for the second order are determined dynamically as updated information on demand and inventory is processed.

2.1.3. Contribution of this Chapter

In this chapter we focus on elucidating the structure of optimal policies. Since our representation of the problem allows for both backorders and lost sales, it provides an integrated view of the growing literature on the newsvendor models with a second order opportunity. In particular, Model I generalizes the seminal model of Barankin (1961); Model II solves a generalized version of the problem considered by Fisher et al. (2001); and, Model III establishes for the first time the structure of the optimal policy for the ordering time and quantity in a dynamic second order decision problem. Our modeling framework nicely complements the recent work of Milner and Kouvelis (2002) whose terminology, (static, partially dynamic and fully dynamic,) we have adopted. The key differences are that 1) they do not allow for partial backorders; and, 2) they focus on examining the interplay between the value of information and flexibility in their three problem variants, while we focus on elucidating the problem structure under rather general conditions.

2.1.4. Organization of this Chapter

The remainder of the chapter is organized as follows. In Section 2, we present the formulation and analysis for Model I. Followed in Sections 3 and 4 are, respectively, the formulation and analysis for Model II and Model III. Related literature is reviewed at the end of each modeling section. This chapter ends with some conclusions in Section 5.

2.2. Model I: The Static Case

2.2.1. Model Description

We develop a discrete time model of the selling season that consists of T periods numbered $1,2,\cdots,T$. The demand in each period is random and independent across periods. At the beginning of period 1 (or at the start of the season), the decision maker places two orders. The first order is delivered before demand is realized in period 1 and the second order is delivered at the beginning of period $L+1$ (after demand is realized in the first L periods). Demand in excess of the first order is accepted to be filled from the second order as long as sufficient stock remains. Demand that cannot be filled from the second order, is considered lost, or effectively, satisfied from outside the system. Any leftover inventory at the end of the season is disposed off at a known cost. The decision maker must choose these two order quantities to minimize the total expected cost of meeting the demand. This cost is the sum of the purchase cost of the two orders and the expected cost of backorders, lost sales and leftover inventory disposal at the end of period T.

2.2.2. Notation

Before formulating the problem it is convenient to define the following notation and terms.

Decision variables

Q_0: The quantity delivered at the beginning of period 1

Q_1: The quantity delivered at the beginning of period $L+1$

$IP_1 = Q_0 + Q_1$: The inventory position at the beginning of period 1

Economic parameters

c_0: The unit purchase cost for Q_0

c_1: The unit purchase cost for Q_1

c_d: The unit disposal cost charged at the end of the season

c_u: The unit lost-sales cost

b_t: The unit backorder cost charged on cumulative backorders in period t; $t = 1,\cdots,T$

Demand variables

ξ_t: The random demand in period t with probability density function (PDF) $f_t(\cdot)$, cumulative distribution function (CDF) $F_t(\cdot)$, and complementary CDF $\overline{F}_t(\cdot)$

ξ_{t_1,t_2}: The cumulative demand in periods t_1 through t_2; $1 \le t_1 \le t_2 \le T$

$$\xi_{t_1,t_2} = \xi_{t_1} + \xi_{t_1+1} + \cdots + \xi_{t_2}$$

$f_{t_1,t_2}(\cdot)$, $F_{t_1,t_2}(\cdot)$, $\overline{F}_{t_1,t_2}(\cdot)$: The PDF, CDF and complementary CDF for ξ_{t_1,t_2}, respectively

State variables and their dynamics

x_t: The inventory level at the beginning of period t; $t = 1,\cdots,T$

y_t: The inventory position (x_t + on-order in period t) at the beginning of period t; $t = 1,\cdots,T$

These state variables evolve as follows.

$$x_1 = Q_0, x_t = \begin{cases} x_{t-1} - \xi_{t-1} & t = 2,\cdots,L \\ x_{t-1} - \xi_{t-1} + Q_1 & t = L+1 \\ x_{t-1} - \xi_{t-1} & t = L+2,\cdots,T \end{cases}$$

$$y_t = \begin{cases} x_t + Q_1 & t = 1,\cdots,L \\ x_t & t = L+1,\cdots,T \end{cases}$$

Note that $y_t \ge x_t$.

Performance measures

$B_t(x_t, y_t \mid \xi_t)$: The realized backorder cost in period t with a beginning inventory level x_t and a beginning inventory position y_t for a given demand ξ_t; $t = 1,\cdots,T$

$$B_t(x_t, y_t \mid \xi_t) = \begin{cases} 0 & \text{If } 0 \le \xi_t < x_t \\ b_t(\xi_t - x_t) & \text{If } x_t \le \xi_t < y_t \\ b_t(y_t - x_t) & \text{If } y_t \le \xi_t \end{cases}$$

$B_t(x_t, y_t)$: The expected backorder cost in period t; $t = 1, \cdots, T$

$$\begin{aligned} B_t(x_t, y_t) &= \int_0^\infty B_t(x_t, y_t \mid \xi_t) f_t(\xi_t) d\xi_t \\ &= b_t \int_{x_t}^{y_t} (\xi_t - x_t) f_t(\xi_t) d\xi_t + b_t \int_{y_t}^\infty (y_t - x_t) f_t(\xi_t) d\xi_t \\ &= b_t\left(B_t^y(y_t) - B_t^x(x_t)\right) \quad \text{where} \end{aligned} \quad \text{... (1)}$$

$$B_t^x(x_t) = \int_0^{x_t} \xi_t f_t(\xi_t) d\xi_t + \int_{x_t}^\infty x_t f_t(\xi_t) d\xi_t$$

$$B_t^y(y_t) = \int_0^{y_t} \xi_t f_t(\xi_t) d\xi_t + \int_{y_t}^\infty y_t f_t(\xi_t) d\xi_t$$

$G_{L+1,T}(y_{L+1} \mid \xi_{L+1,T})$: The realized total cost of lost sales and disposal with a beginning inventory level y_{L+1} for a given demand $\xi_{L+1,T}$ (Note that $y_{L+1} = x_{L+1}$ due to the delivery of Q_1)

$$G_{L+1,T}(y_{L+1} \mid \xi_{L+1,T}) = \begin{cases} c_d(y_{L+1} - \xi_{L+1,T}) & \text{If } \xi_{L+1,T} < y_{L+1} \\ c_u(\xi_{L+1,T} - y_{L+1}) & \text{If } \xi_{L+1,T} \ge y_{L+1} \end{cases}$$

$G_{L+1,T}(y_{L+1})$: The expected total cost of lost sales and disposal with a beginning inventory level y_{L+1}

$$G_{L+1,T}(y_{L+1}) = \int_0^\infty G_{L+1,T}(y_{L+1} \mid \xi_{L+1,T}) f_{L+1,T}(\xi_{L+1,T}) d\xi_{L+1,T}$$

$g_t(x_t, y_t)$: The expected total cost of backorders in periods t through T, lost sales and disposal with a beginning inventory level x_t and a beginning inventory position y_t; $t = 1, \cdots, L$

$$g_t(x_t, y_t) = \begin{cases} B_t(x_t, y_t) + \int_0^\infty g_{t+1}(x_t - \xi_t, y_t - \xi_t) f_t(\xi_t) d\xi_t & t = 1, \cdots, L-1 \\ B_L(x_L, y_L) + \int_0^\infty G_{L+1,T}(y_L - \xi_L) f_L(\xi_L) d\xi_L & t = L \end{cases} \quad \text{... (2)}$$

$TC_I(Q_0)$: The minimum expected total cost of the T-period problem with an initial order Q_0

2.2.3. Model Formulation

We are now ready to develop Model I. The decision problem for IP_1 at the beginning of period 1, which essentially determines the second order quantity $Q_1 = IP_1 - Q_0$, can be formulated as

$$h_1(x_1) = \min_{IP_1 \geq x_1} \{H_1(x_1, IP_1) = c_1(IP_1 - x_1) + g_1(x_1, IP_1)\} \quad \text{... (P1.1)}$$

The first term of $H_1(x_1, IP_1)$ represents the purchase cost of the second order, which is conveniently defined in terms of the inventory position; and the second term represents the operating cost in periods 1 through T for a beginning inventory level x_1 and a beginning inventory position IP_1. Similarly, the decision problem for Q_0 prior to the start of the season can be formulated as

$$\min_{Q_0 \geq 0} \{TC_I(Q_0) = c_0 Q_0 + h_1(x_1 = Q_0)\} \quad \text{... (P1.2)}$$

The first term of $TC_I(Q_0)$ represents the purchase cost of the first order; and the second term represents the minimum expected total cost in periods 1 through T with a beginning inventory level Q_0, excluding the purchase cost for Q_0. We let $IP_1^*(x_1)$ be the optimal solution of (P1.1) and Q_0^* be the optimal solution of (P1.2).

2.2.4. Analysis and Results

Since the first term in (P1.2) is linear, to understand the structure of problem (P1.2), it is sufficient to focus our analysis on the structure of $h_1(x_1)$ in (P1.1), which in turn requires an analysis of $H_1(x_1, IP_1)$. Since $H_1(x_1, IP_1)$ is determined by the expected backorder costs in periods 1 through L via $g_t(x_t, y_t)$ and by $G_{L+1,T}(y_{L+1})$, we begin our analysis by characterizing $g_t(x_t, y_t)$, $G_{L+1,T}(y_{L+1})$, $B_t^x(x_t)$ and $B_t^y(y_t)$. (All proofs for Lemmas and Theorems below for this chapter are provided in Appendix A.)

Lemma 1.

1) $B_t^x(x_t)$ is concave increasing in x_t; and, $B_t^y(y_t)$ is concave increasing in y_t.
2) $G_{L+1,T}(y_{L+1})$ is convex in y_{L+1}.
3) $g_1(x_1, y_1)$ is separable in x_1 and y_1. Moreover, $g_1(x_1, y_1)$ is convex in x_1.

It is important to notice that $B_t(x_t, y_t)$ from (1) has the special structure that, since it is a linear function of the difference of $B_t^y(y_t)$ and $B_t^x(x_t)$, it is separable in x_t and y_t. An intuitive explanation follows by observing that $B_t^y(y_t)$ represents the expected sales, including backorders, in period t that are satisfied by the inventory position y_t; and, $B_t^x(x_t)$ represents the expected sales that are satisfied in period t from the on-hand inventory x_t. Hence, the expected demand backordered in period t is the non-negative difference of these two quantities, each of which depends on one variable only. An important consequence is that $g_1(x_1, y_1)$ is separable in x_1 and y_1.

Notice that $B_t(x_t, y_t)$ is a "cost" that is the difference of two concave functions, a fact that significantly complicates the analysis of our problem.

However, $G_{L+1,T}(y_{L+1})$ is convex in the argument which significantly facilitates the analysis. Now, consider the first order optimality condition of $H_1(x_1, IP_1)$ in (P1.1) with respect to IP_1, which is given by

$$\frac{\partial H_1(x_1, IP_1)}{\partial IP_1} = c_1 + \frac{\partial g_1(x_1, IP_1)}{\partial IP_1} \qquad \text{... (3)}$$

Or, equivalently,

$$\frac{\partial H_1(x_1, IP_1)}{\partial IP_1} = (c_1 + c_d) + \sum_{t=1}^{L} b_t \bar{F}_{1,t}(IP_1) - (c_u + c_d)\bar{F}_{1,T}(IP_1) = 0 \qquad \text{... (4)}$$

The second order derivative is $-\sum_{t=1}^{L} b_t f_{1,t}(IP_1) + (c_u + c_d) f_{1,T}(IP_1)$. As can be seen the second order derivative is not clearly positive or negative, suggesting that $H_1(x_1, IP_1)$ is in general not convex, or even unimodal in IP_1, making intractable a full characterization. However, there is sufficient structure to come close to a complete characterization. Since we know from (4) that $H_1(x_1, IP_1)$ as a function of IP_1 may have multiple minima and maxima, the order-up-to position IP_1^* depends on x_1; and since x_1 is a non-negative state variable, IP_1^* must be defined for all possible non-negative realizations of x_1. We proceed iteratively starting at $x_1 = 0$. To initialize set $IP_1^0 = 0$. Now iteratively define IP_1^i as the largest local minimum that is greater than IP_1^{i-1} and yields the lowest cost. Note that under this procedure, as i increases, the target IP_1^i increases, possibly skipping smaller local optima that have higher expected cost. Suppose that this procedure yields "n" minima for $H_1(x_1, IP_1)$ denoted by $IP_1^1 < IP_1^2 < \cdots < IP_1^n$. Each of these IP_1 values represents the target inventory position given that an order is placed and x_1 is positive. Now notice that in each interval $\left[IP_1^{i-1}, IP_1^i\right]$ there exists a unique value for x_1, denoted by a^i, such that $H_1(x_1, x_1) < H_1(x_1, IP_1^i)$ for $IP_1^{i-1} \le x_1 < a^i$ and $H_1(x_1, x_1) \ge H_1(x_1, IP_1^i)$ for $a^i \le x_1 \le IP_1^i$. Hence, in the interval

$\left[IP_1^{i-1}, a^i\right)$ it is optimal not to place an order, and it is optimal to place an order $Q_1 = IP_1^i - x_1$ in the interval $\left[a^i, IP_1^i\right]$. Formally, the optimal second order quantity is given by

$$Q_1^* = \begin{cases} 0 & \text{If } IP_1^0 \le x_1 < a^1 \\ 0 & \text{If } IP_1^{i-1} < x_1 < a^i \quad i = 2, \cdots, n \\ IP_1^i - x_1 & \text{If } a^i \le x_1 \le IP_1^i \quad i = 1, \cdots, n \\ 0 & \text{If } IP_1^n < x_1 \end{cases} \qquad \dots (5)$$

It follows from this policy that $h_1(x_1)$ is piecewise convex. To solve (P1.2), rather than analyze $h_1(x_1)$ directly, we approach the problem by first considering the two functions introduced below.

$$j_1(x_1) = \begin{cases} H_1\left(x_1, IP_1^1\right) & \text{If } IP_1^0 \le x_1 \le IP_1^1 \\ H_1\left(x_1, IP_1^i\right) & \text{If } IP_1^{i-1} < x_1 \le IP_1^i \quad i = 2, \cdots, n \\ H_1(x_1, x_1) & \text{If } IP_1^n < x_1 \end{cases}$$

$$k_1(x_1) = H_1(x_1, x_1)$$

It can be seen that $h_1(x) = \min\{j_1(x_1), k_1(x_1)\}$ and it will yield the optimal policy defined in (5) above for the second order quantity. Except in the boundary case where $j_1(x_1)$ and $k_1(x_1)$ are expediently set to be identically equal, $k_1(x_1)$ represents the expected cost if the second order is not placed, and, $j_1(x_1)$ represents the minimum expected cost given that an order must be placed to bring the inventory position to the IP_1^i not less than and closest to x_1. To proceed we need the properties of $j_1(x_1)$ and $k_1(x_1)$ given below.

Lemma 2.

1) $j_1(x_1)$ is piecewise convex in x_1, and $k_1(x_1)$ is convex in x_1.

2) $\frac{dj_1(x_1)}{dx_1} = \frac{dk_1(x_1)}{dx_1}$ at all $x_1 = IP_1^i \, (i \neq 0)$.

3) $\frac{dj_1(x_1)}{dx_1}$ is continuous and increasing in x_1.

Since $h_1(x) = \min\{j_1(x_1), k_1(x_1)\}$, it follows by (P1.2) that

$$TC_I(Q_0) = \min\{c_0Q_0 + j_1(Q_0), c_0Q_0 + k_1(Q_0)\} \qquad \ldots (6)$$

The advantage of representing $TC_I(Q_0)$ by (6) is that we can exploit the structure of $j_1(x_1)$ and $k_1(x_1)$. Let Q_0^{**} be the maximum between zero and the value of x_1 that satisfies $\frac{dj_1(x_1)}{dx_1} = -c_0$, and Q_0^{***} be the maximum between zero and the value of x_1 that satisfies $\frac{dk_1(x_1)}{dx_1} = -c_0$. Interestingly, because of Lemma 2, we can see Q_0^{**} and Q_0^{***} fall in the same interval. This is because, it can be shown that $\left.\frac{dj_1(x_1)}{dx_1}\right|_{IP_1^i} = \left.\frac{dk_1(x_1)}{dx_1}\right|_{IP_1^i} < \left.\frac{dj_1(x_1)}{dx_1}\right|_{IP_1^{i+1}} = \left.\frac{dk_1(x_1)}{dx_1}\right|_{IP_1^{i+1}}$, and that $j_1(x_1)$ and $k_1(x_1)$ are equal at each breakpoint IP_1^i. Hence $c_0x_1 + j_1(x_1)$ and $c_0x_1 + k_1(x_1)$ are minimized in the same interval $IP_1^{l-1} \leq x_1 \leq IP_1^l$ for some l, so that the solution at the breakpoint IP_1^i other than Q_0^{**} and Q_0^{***} can not be optimal. As a result only $c_0Q_0^{***} + k_1(Q_0^{***})$ and $c_0Q_0^{**} + j_1(Q_0^{**})$ have to be compared to determine the initial order quantity, formalized in the next theorem.

Theorem 1. The optimal solution $Q_0^* = \arg\min\left\{c_0Q_0^{***} + k_1\left(Q_0^{***}\right), c_0Q_0^{**} + j_1\left(Q_0^{**}\right)\right\}$.

While Q_0^{***}, $k_1\left(Q_0^{***}\right)$ and Q_0^{**} are easy to compute, calculating $j_1\left(Q_0^{**}\right)$ can be difficult because it requires selecting the appropriate breakpoint IP_1^i. Hence, in general a line search would be required to obtain all admissible breakpoints before the choice between them can be made. Alternatively, we may put restrictions on the specification of the problem to guarantee that $H_1\left(x_1, IP_1\right)$ is sufficiently well behaved in IP_1 to yield one local minimum. The latter approach yields the following Theorem 2. For this theorem and some other results in the thesis, we need PF_2 density for demand. We state a key property of PF_2.

Variation Diminishing Property (VDP) of PF_2

Let $M(u)$ be a real function on $(-\infty,\infty)$, f be PF_2 on $[0,\infty)$ and zero on $(-\infty,0]$. Suppose $M(u)$ changes sign at most once on $(-\infty,\infty)$. Then the transformation

$$g(y) = \int_{-\infty}^{\infty} M(u) f(y-u) du$$

also changes sign at most once over $(-\infty,\infty)$. Moreover, if both g and M change sign once, their sign changes occur in the same order (see Karlin 1968).

Theorem 2.

1) If $L=0$; equivalently, $L>0, b_1=\cdots=b_L=0$, then $H_1\left(x_1, IP_1\right)$ is convex in IP_1.

2) If $L=T, b_1=\cdots=b_{T-1}=0$ and $b_T>0$, then $H_1\left(x_1, IP_1\right)$ is convex in IP_1.

3) If $L>0$, $b_1=\cdots=b_{L-1}=0, b_L>0$ and the monotone likelihood ratio property

(MLRP) holds for $\frac{f_{1,T}(IP_1)}{f_{1,L}(IP_1)}$ over $[0,\infty)$, then $H_1(x_1, IP_1)$ has at most one local maximum and one local minimum with respect to IP_1.

4) If $c_1 + \sum_{l=1}^{L} b_l - c_u \leq 0$ and $\xi_1, \cdots, \xi_L$ have independent PF_2 densities, then $H_1(x_1, IP_1)$ is unimodal in IP_1.

5) If $\xi_1, \cdots, \xi_L$ have independent PF_3 densities, then $H_1(x_1, IP_1)$ has at most one local minimum and one local maximum with respect to IP_1.

2.2.5. Related Literature

Theorem 2 identifies five specifications each of which guarantees that either (3) has a unique root which is the only minimum, or $H_1(x_1, IP_1)$ is increasing in IP_1. Interestingly, the antecedents of Theorem 2 part 2) can be traced to the work of Barankin (1961). In his model an "emergency" order is placed and delivered at the beginning of the season, while the second arrives at the end of the season, so that L is effectively equal to T. When this is the case it is easy to verify that under the conditions of part 2) the problem remains separable, and that it is convex in each of the two variables. This is also the case in the more recent two-order-opportunity model of demand management due to Iyer et al. (2003), which is directly related to the work by Barankin. In this model, an initial order, corresponding to Q_0, is placed for delivery at the start of the season. Subsequently, there is an information phase in which the demand forecast for the season is revised. This revised information is considered in placing the second order, Q_1. And, this information is concurrently used to split demand into two groups, that which will be satisfied from Q_0, and the rest that will be satisfied from Q_1, and possibly, the unused portion of Q_0. In this sense, this model also has an effective leadtime of $L = T$. As will be discussed in the next section, incorporating forecast revision, based on information from the pre-season, can be easily accommodated in our construct of the problem.

Exploiting the benefit of updated information from the pre-season also underlies the work of Donohue (2000), whose model is a one-product variant of the model due to Fisher and Raman (1996); a similar model is also considered by Jones et al. (2001). In these models $L=0$, so that demand is either satisfied from inventory on-hand or it is lost. Hence, as articulated in Theorem 2 part 1), this yields the convexity in $IP_1 = Q_0 + Q_1$.

The last three parts of Theorem 2, accommodate the case when L is between 0 and T; but place mild but increasingly more restrictive regularity conditions as the modeling environment is enriched. Part 3) addresses those scenarios in which backorder costs are not time dependent; the densities must exhibit the frequently used monotone likelihood ratio property (Karlin and Rubin 1956). When backorder costs are time dependent, part 4), accommodates the case when it is always more economical to use a unit from the second order to meet a backorder than to let it be lost; but the densities must exhibit the specific variation diminishing property of the PF_2 class. Analogously, under arbitrary economic parameters, the densities must exhibit the more restrictive variation diminishing property of the PF_3 class, which is a subset of the PF_2 class. The development of PF_n densities is due to Schoenberg (1951); and, its initial application to stochastic inventory models appears to be due to Karlin (1958, 1968). Recently Porteus (2002), provides an intuitive explanation of this family of densities. Importantly, PF_2 densities have the monotone likelihood ratio property, and PF_3, PF_2 and MLRP densities include the classes of uniform, truncated normal and gamma families.

2.2.6. Summary

To summarize, in this section we have developed a static version of the two-order-opportunity newsvendor model. Our key modeling innovation is that we allow for partial backorders, whose cost may be time-dependent, during the season until the second order is received. Once demand exceeds total stock, demand is lost. Our technical innovation is to observe that the resulting two-variable optimization problem is separable in its arguments. This allows us to develop an expedient sequential solution procedure, which can be naturally applied to the next two models. Since the problem need not be well behaved with respect to the second order decision, we develop and present mild regularity conditions that guarantee that the optimization problem becomes well behaved in the sense that it admits only one solution.

In the next two sections, we will use the static case as a crucial building block for more dynamic versions of the problems. Such problems have received significant attention in the recent literature, primarily because of the connection of such models to quick response practices.

2.3. Model II: The Partially Dynamic Case

As we discussed at the end of the last section, Model I can accommodate those settings in which information is updated between the placement of the initial order and the start of the season. Consequently, the second order is based on an updated understanding of the degree of uncertainty faced by the decision maker. Being able to defer, albeit at higher unit cost, the decision to place the second order until after the start of the season provides additional flexibility. It has the additional independent advantage of making up for faster rate of inventory depletion than anticipated when the first order was placed. It is the goal of this section to study the deferment of the second order, so that its quantity is dynamically adjusted for depletion of stock and enhanced by the possible availability of more refined forecasts of demand.

2.3.1. Model Description

This adaptation leads to the following generalization of Model I. In Model II, the second order, if there is one, is placed at the beginning of period τ so that it is delivered at the beginning of period $\tau + L(\le T+1)$. Consequently, in the event that during the first $\tau - 1$ periods the initial inventory, Q_0, is depleted, we continue to accept all additional demand as backorders for the guaranteed delivery at the beginning of period $\tau + L$. At the beginning of period τ, after observing the inventory level, x_τ, if necessary, we place an order of size $Q_\tau = IP_\tau - x_\tau$, to bring the inventory position to its desired target level IP_τ. As in Model I, during periods τ through $\tau + L - 1$, we only accept cumulative demand in excess of $\max\{x_\tau, 0\}$ if it does not exceed IP_τ. And, as in Model I, after the second order is depleted no backorders are accepted.

2.3.2. Notation

Since, the dynamics of Model II from periods τ to T are analogous to those of Model I, its notation and analysis can be readily adapted to facilitate the formulation and analysis of Model II. As in Model I, characterizing the optimal IP_τ can pose computational challenges. However, for Model II, under mild regularity conditions like those of Theorem 2 for Model I, determining the optimal IP_τ becomes a well-behaved optimization problem. To proceed we need to amend and append the following notation.

Q_τ: The second order placed at the beginning of period τ for delivery at the beginning of period $\tau + L$

$IP_\tau = x_\tau + Q_\tau$: The inventory position at the beginning of period τ after the second order is placed

State variables evolve as follows.

$$x_1 = Q_0, x_t = \begin{cases} x_{t-1} - \xi_{t-1} & t = 2, \cdots, \tau + L - 1, \tau + L + 1, \cdots, T \\ x_{t-1} - \xi_{t-1} + Q_\tau & t = \tau + L \end{cases}$$

$$y_t = \begin{cases} x_t & t = 1, \cdots, \tau - 1 \\ x_t + Q_\tau & t = \tau \\ y_{t-1} - \xi_{t-1} & t = \tau + 1, \cdots, T \end{cases}$$

$g_\tau(x_\tau, y_\tau)$: The expected total cost of backorders in periods τ through T, lost sales and disposals with a beginning inventory level x_τ and a beginning inventory position y_τ (analogous to $g_1(x_1, y_1)$ in Model I)

$h_t(x_t)$: The minimum expected total cost of purchase for the second order, backorders in periods t through T, lost sales and disposals with a beginning inventory level x_t; $1 \le t \le \tau$

$$h_t(x_t) = b_t \int_{x_t}^{\infty} (\xi_t - x_t) f_t(\xi_t) d\xi_t + \int_0^{\infty} h_{t+1}(x_t - \xi_t) f_t(\xi_t) d\xi_t \text{ for } 1 \le t \le \tau - 1 \qquad \dots (7)$$

$TC_{II}(Q_0)$: The minimum expected total cost of the T-period problem with an initial order Q_0

2.3.3. Model Formulation

We are now ready to develop Model II. The decision problem at the beginning of period τ, analogous to (P1.1) in Model I, can be formulated as

$$h_\tau(x_\tau) = \min_{IP_\tau \ge x_\tau} \{ H_\tau(x_\tau, IP_\tau) = c_\tau(IP_\tau - x_\tau) + g_\tau(x_\tau, IP_\tau) \} \qquad \dots (P2.1)$$

The first term of $H_\tau(x_\tau, IP_\tau)$ represents the purchase cost of the second order, which is conveniently defined in terms of the inventory position; and the second term represents the operating cost in periods τ through T for a beginning inventory level x_τ and a beginning inventory position IP_τ, exclusive of the purchase cost for $Q_\tau = IP_\tau - x_\tau$.

Similarly, the decision problem for Q_0 prior to the start of the season can be formulated as

$$\min_{Q_0 \geq 0} \left\{ TC_{II}(Q_0) = c_0 Q_0 + h_1(x_1 = Q_0) \right\} \qquad \text{... (P2.2)}$$

The first term of $TC_{II}(Q_0)$ represents the purchase cost of the first order; and the second term represents the minimum expected total cost in periods 1 through T with a beginning inventory level Q_0, exclusive of the purchase cost for Q_0. We let $IP_\tau^*(x_\tau)$ be the optimal solution of (P2.1) and Q_0^* be the optimal solution of (P2.2).

2.3.4. Analysis and Results

Since by (7) $h_1(x_1)$ in (P2.2) is essentially determined by $h_\tau(x_\tau)$ from (P2.1), we begin with its analysis. While $h_\tau(x_\tau)$ is similar to $h_1(x_1)$ in (P1.1), the significant difference is that while in Model I the starting value of x_1 could not be negative, that need not be the case here. This is because x_τ is now a state variable that can take negative realizations. Consequently, when x_τ is negative, at least for those customers whose demand was backordered, we must order a minimum of $-x_\tau$ units at a unit cost $c_\tau + \sum_{t=\tau}^{\tau+L-1} b_t$. Hence, after a decision is made at time τ, the inventory position is non-negative. The definition of IP_τ^0 must be slightly modified from IP_1^0 in Model I, for those instances when x_τ is non-positive and $H_\tau(0,0) > H_\tau(0, IP_\tau^1)$ holds. If $H_\tau(0,0) \leq H_\tau(0, IP_\tau^1)$, IP_τ^0 is still zero; However, if $H_\tau(0,0) > H_\tau(0, IP_\tau^1)$, it is optimal to bring the inventory level to IP_τ^1.

To accommodate this contingency reset IP_τ^0 to $-\infty$ and define a^1 as $-\infty$. Now notice that when IP_τ^{i-1} is finite, in each interval $\left[IP_\tau^{i-1}, IP_\tau^i\right]$ there exists a unique value for x_τ, denoted by a^i, such that $H_\tau(x_\tau, x_\tau) < H_\tau\left(x_\tau, IP_\tau^i\right)$ for $IP_\tau^{i-1} < x_\tau < a^i$ and $H_\tau(x_\tau, x_\tau) \geq H_\tau\left(x_\tau, IP_\tau^i\right)$ for $a^i \leq x_\tau \leq IP_\tau^i$. Hence, in the interval $\left(IP_\tau^{i-1}, a^i\right)$ it is optimal not to place an order, and it is optimal to place an order $Q_\tau = IP_\tau^i - x_\tau$ in the interval $\left[a^i, IP_\tau^i\right]$. Formally, the optimal second order quantity, analogous to (5), is given by

$$Q_\tau^{**} = \begin{cases} -x_\tau & \text{If } x_\tau \leq IP_\tau^0 \\ 0 & \text{If } IP_\tau^{i-1} < x_\tau \leq a^i \quad i = 1, \cdots, n \\ IP_\tau^i - x_\tau & \text{If } a^i < x_\tau \leq IP_\tau^i \quad i = 1, \cdots, n \\ 0 & \text{If } IP_\tau^n < x_\tau \end{cases} \quad \ldots (8)$$

Hence, we must define $j_\tau(x_\tau)$ and $k_\tau(x_\tau)$, analogous to $j_1(x_1)$ and $k_1(x_1)$ in Model I, as

$$j_\tau(x_\tau) = \begin{cases} H_\tau(x_\tau, 0) & \text{If } x_\tau \leq IP_\tau^0 \\ H_\tau\left(x_\tau, IP_\tau^i\right) & \text{If } IP_\tau^{i-1} < x_\tau \leq IP_\tau^i \quad i = 1, \cdots, n \\ H_\tau(x_\tau, x_\tau) & \text{If } IP_\tau^n < x_\tau \end{cases} \quad \ldots (9)$$

$$k_\tau(x_\tau) = \begin{cases} H_\tau(x_\tau, 0) & \text{If } x_\tau \leq IP_\tau^0 \\ H_\tau(x_\tau, x_\tau) & \text{If } x_\tau > IP_\tau^0 \end{cases}$$

It can be seen that $h_\tau(x_\tau) = \min\left\{j_\tau(x_\tau), k_\tau(x_\tau)\right\}$ and it will yield the optimal policy defined in (8). Except in the boundary cases where $j_\tau(x_\tau)$ and $k_\tau(x_\tau)$ are expediently set to be identically equal, $k_\tau(x_\tau)$ represents the expected cost if the second order is not placed, and, $j_\tau(x_\tau)$ represents the minimum expected cost given that an order must be placed to bring the inventory position to the smallest IP_τ^i not less than x_τ.

In a deviation from Model I, we then precede recursively to define

$$j_1(x_1)=\sum_{t=1}^{\tau-1} b_t \int_{x_1}^{\infty}(\xi_{1,t}-x_1)f_{1,t}(\xi_{1,t})d\xi_{1,t}+\int_0^{\infty} j_\tau(x_1-\xi_{1,\tau-1})f_{1,\tau-1}(\xi_{1,\tau-1})d\xi_{1,\tau-1}$$

$$k_1(x_1)=\sum_{t=1}^{\tau-1} b_t \int_{x_1}^{\infty}(\xi_{1,t}-x_1)f_{1,t}(\xi_{1,t})d\xi_{1,t}+\int_0^{\infty} k_\tau(x_1-\xi_{1,\tau-1})f_{1,\tau-1}(\xi_{1,\tau-1})d\xi_{1,\tau-1}$$

Here, the first term for each of $j_1(x_1)$ and $k_1(x_1)$ represents, the expected backorder cost in periods 1 through $\tau-1$. Thus, it follows that

$$TC_{II}(Q_0)=\min\{c_0Q_0+j_1(Q_0),c_0Q_0+k_1(Q_0)\} \quad \text{... (10)}$$

whose derivation has taken advantage of (7) and $h_\tau(x_\tau)=\min\{j_\tau(x_\tau),k_\tau(x_\tau)\}$. While $h_\tau(x_\tau)$ in (P2.1) inherits all the properties of $h_1(x_1)$ in (P1.1), it is unfortunately not the case for $j_1(x_1)$ of Model II: We are not able to develop results similar to that in Lemma 2 to facilitate the solution for (10); This is because that the discontinuities that arises from (9) when $n>1$ can not be smoothed away. Hence, being able to guarantee that an efficient computational scheme can be devised for finding Q_0^* requires imposing mild regularity conditions that guarantee the existence of no more than one local minimum. This results in the following theorem that is analogous to Theorem 2.

Theorem 3.

1) If $L=0$; equivalently, $L>0, b_\tau=\cdots=b_{\tau+L-1}=0$, then $H_\tau(x_\tau,IP_\tau)$ is convex in IP_τ and $TC_{II}(Q_0)$ is convex in Q_0.

2) If $L = T - \tau + 1, b_\tau = \cdots = b_{\tau+L-2} = 0$ and $b_{\tau+L-1} > 0$, then $H_\tau(x_\tau, IP_\tau)$ is convex in IP_τ and $TC_{II}(Q_0)$ is convex in Q_0.

3) If $L > 0$, $b_\tau = \cdots = b_{\tau+L-2} = 0, b_{\tau+L-1} > 0$ and the monotone likelihood ratio property (MLRP) holds $\frac{f_{\tau,T}(IP_1)}{f_{\tau,L}(IP_1)}$ over $[0,\infty)$, then $H_\tau(x_\tau, IP_\tau)$ has at most one local maximum and one local minimum with respect to IP_τ.

4) If $c_\tau + \sum_{t=\tau}^{\tau+L-1} b_t - c_u \le 0$ and $\xi_\tau, \cdots, \xi_{\tau+L-1}$ have independent PF_2 densities, then $H_\tau(x_\tau, IP_\tau)$ is unimodal in IP_τ and $TC_{II}(Q_0)$ is convex in Q_0.

5) If $\xi_\tau, \cdots, \xi_{\tau+L-1}$ have independent PF_3 densities, then $H_\tau(x_\tau, IP_\tau)$ has at most one local minimum and one local maximum with respect to IP_τ.

Notice that in part 2) of the theorem, to assure that the second order is delivered at the end of the season we have slightly redefined L. Moreover, whenever $H_\tau(x_\tau, IP_\tau)$ is unimodal in IP_τ, $h_\tau(x_\tau)$ is convex. This convexity facilitates its adaptation to incorporate forecast revision, as discussed next.

2.3.5. Forecast Revision

While we have formulated both Model I and Model II without allowing for forecast revision, the models can easily accommodate forecast revision in the pre-season as well as additional updates based on observed demands under mild regularity conditions. Since Model I can be viewed as a special case of Model II with $\tau = 1$, it is sufficient to focus on incorporating forecast revision into Model II. Let S be the vector of information from the pre-season and from demand during periods 1 through $\tau - 1$ that is equivalent to a sufficient statistic available at the beginning of period τ; this statistic is used to determine posterior distributions of demand. The key to the analysis is to recognize that if $h_\tau(x_\tau)$

conditional on the sufficient statistic is convex, then under the expectation over the distribution of S, it will remain convex and therefore $h_1(x_1)$ is convex by (7) (see Scarf 1959). Under the conditions of Theorem 2 parts 1) and 2), the convexity of $h_\tau(x_\tau)$ is always assured so any updating method can be used. While for part 4), we must assure that the revised distributions remain in the PF_2 class. This is guaranteed if forecast revision is executed using conjugate priors (See, for example, Berger 1985).

2.3.6. Related Literature

It is important to notice that Model II is a generalization of the model considered by Fisher et al. (2001), that allows c_0 to be different than c_τ and allows the backorder cost during the leadtime to be time-dependent. Fisher et al. focused on developing algorithmic solutions to the problem because they observed that this problem is not convex in the two decision variables. In contrast, we have shown that under mild regularity conditions that accommodate the case of normal demand considered by them, the problem can be reduced to a convex optimization problem in one variable.

While Model II allows for backorders of all unfilled demand until the second order is depleted, Eppen and Iyer (1997) considered the version of the problem in which demand must be filled only by the inventory on hand. Moreover, during the pre-season, the supplier agrees to a total maximal commitment that is proportional to the initial order quantity. An initial amount is chosen before the start of the season, and the second order, up to the capacity limit, is placed at the beginning of period τ with zero leadtime; the second quantity is based on an revised demand forecast, while allowing for returns from customers, an important feature of the catalog sales company that provided the motivation for that study. It is proven that this optimization problem is convex. The partially dynamic model of Milner and Kouvelis (2002), for the case of demand described by Brownian motion, directly generalizes the essential version of the model of Eppen and Iyer by making the leadtime positive.

2.3.7. Summary

To summarize, we have shown that solving Model II reduces to sequentially determining the initial order quantity after determining the second order quantity for all possible states of x_τ at the beginning of period τ. If finding the target level entails minimizing a unimodal function, determining the initial order quantity reduces to finding the unique minimum of a convex function. Moreover under the same regularity conditions both Model I and Model II can be generalized to incorporate forecast revisions. Now that we have shown that Model I and II with and without forecast revision encompass many variants of the recent and early literature on newsvendor models with two purchase opportunities, we consider the final variant of the model in which both the second order quantity and its timing are determined dynamically.

2.4. Model III: The Fully Dynamic Case

Using results from the previous two sections as building blocks, we consider the most general setting for the newsvendor model with a second order opportunity. Not only does this preserve the flexibility of determining the second order quantity dynamically, it offers the additional flexibility of determining dynamically when the second order is placed. Hence, if demand is higher than anticipated when the first order was placed, it is likely that the second order will be placed earlier in the season for a larger quantity. In contrast, if demand is lower than anticipated, the second order will be placed later in the season for a smaller quantity. It is the goal of this section, to formulate and analyze the structure of this problem.

2.4.1. Model Description

To proceed with the generalization, we amend the dynamics of Model II. As in Model II, in Model III we start the season with Q_0 units in stock, so that $x_1 = Q_0$. Then at the beginning of each period $t, t = 1, \cdots, T-L+1$, we determine whether to place the second order at unit cost c_t ($c_t < c_{t+1}$). If the second order is not placed, we accept all demand in period t; that in excess of x_t is backordered for guaranteed delivery at some time in the future. However, if the second order is placed in period t, we continue to accept demand until the inventory position reaches 0, with the understanding that all backorders will be cleared when the second order is received. All subsequent demand is considered lost at a unit penalty cost c_u.

2.4.2. Notation

Since the dynamics of Model III, are similar to those of Models I and II, adapting their notation facilitates the analysis. To proceed we need to amend and append the following notation.

c_t: The unit order cost for the second order if placed at the beginning of period t; $t = 1, \cdots, T-L+1$

$g_t(x_t, y_t)$: The expected cost in periods t through T with a beginning inventory level x_t and a beginning inventory position y_t after the second order is placed at the beginning of period t (analogous to $g_\tau(x_\tau, y_\tau)$ in Model II); $t = 1, \cdots, T-L+1$

$h_t(x_t)$: The minimum expected cost in periods t through T with a beginning inventory level x_t when the second order opportunity is still available, excluding the purchase cost for the first order; $t = 1, \cdots, T-L+1$

$k_t(x_t)$: The minimum expected cost in periods t through T when the second order is not placed at the beginning of period t, including the expected purchase cost for the second order in a future period; $t=1,\cdots,T-L+1$; ($k_{T-L+1}(x_{T-L+1})$ and $j_{T-L+1}(x_{T-L+1})$ are analogous to k_τ and j_τ in Model II, respectively;)

$$k_t(x_t)=b_t\int_{x_t}^{\infty}(\xi_t-x_t)f_t(\xi_t)d\xi_t+\int_0^{\infty}h_{t+1}(x_t-\xi_t)f_t(\xi_t)d\xi_t \quad \text{for } t=1,\cdots,T-L$$

$j_t(x_t)$: The minimum expected cost in periods t through T when the second order is placed at the beginning of period t, including the purchase cost for the second order; $t=1,\cdots,T-L+1$

$$j_t(x_t)=\min_{IP_t\ge x_t}\{H_t(x_t,IP_t)=c_t(IP_t-x_t)+g_t(x_t,IP_t)\} \quad \text{... (11)}$$

2.4.3. Model Formulation

Based on the notation above, we can formulate the decision problem for the second order opportunity at the beginning of period t. Since there are two choices at the beginning of period t: either no order is placed incurring the cost $k_t(x_t)$ or an order is placed incurring the cost $j_t(x_t)$, we have

$$h_t(x_t)=\min\{k_t(x_t),j_t(x_t)\} \quad \text{... (P3.1)}$$

With a beginning inventory level x_t, an order is placed in period t only when $k_t(x_t)\ge j_t(x_t)$.

Similarly, the decision problem for the first order quantity prior to the start of the season is formulated as

$$\min_{Q_0\ge 0}\{TC_{III}(Q_0)=c_0Q_0+h_1(x_1=Q_0)\} \quad \text{... (P3.2)}$$

2.4.4. Analysis and Results

To proceed with the analysis notice that since period $T-L+1$ is the last period when an order can be placed, it follows that the optimal order quantity is given by the equivalent of (8). Hence, it follows that the optimal policy is generally quite complex, so that appealing policies of the form (s,S); that is, order up to S if the inventory level is below s, can not be optimal in general. This is despite the fact that under the conditions of Theorem 3, 2) and 4), $h_t(x_t)$ is convex in x_t, which is sufficient for the optimality of such policies in Model II. Unfortunately, while these conditions are sufficient to guarantee that for small enough x_t, it is optimal to place an order in period t, they are not strong enough to sufficiently smooth the cost function $k_{t+1}(x_{t+1})$ to assure that $h_t(x_t)$ is well-behaved in x_t in general. In particular, we are able to generate robust counter examples with $L>0$ that show that $k_t(IP_t^*)>j_t(IP_t^*)$ (see Table 2-1). As a consequence, in any meaningful policy, $j_t(x_t)$ and $k_t(x_t)$ may cross an even number of times, making (s,S) policies sub-optimal.

Table 2-1 A Counter Example

Parameters: $L=1, T=3,\ c_d=2, c_u=15, c_3=10, b_3=3$, $f_3(x)=\lambda_3 e^{-\lambda_3 x}, \lambda_3=.01, c_2=9,, f_2(x)=\lambda_2 e^{-\lambda_2 x}, \lambda_2=.01$			
b_2	IP_2^*	$j_2(IP_2^*)$	$k_2(IP_2^*)$
3	107.795	1596.17	1624.82
4	100.76	1668.22	1735.46

While we are in general unable to establish that the optimal policy structure is well defined, under appropriate mild regularity conditions the optimal ordering policy is of the (s,S) class as stated in the following theorem.

Theorem 4.

1) If $L=0$ and demand $\xi_1,\cdots\xi_T$ have PF_2 densities, then (s_t,S_t) policy is the optimal decision rule for whether or not to place the second order when the second order opportunity is available.

2) If $L>0, b_1=\cdots=b_{T-1}=0, b_T\geq 0$, and demand $\xi_1,\cdots\xi_T$ have PF_2 densities, then (s_t,S_t) policy is the optimal decision rule for whether or not to place the second order when the second order opportunity is available.

Under conditions of Theorem 4, we can show that in period t $j_t(x_t)\geq k_t(x_t)$ for $x_t\geq IP_t^*$, $j_t(x_t)<k_t(x_t)$ for small enough x_t and that $j_t(x_t)$ and $k_t(x_t)$ cross only once. The intersection point of $j_t(x_t)$ and $k_t(x_t)$ is the order-triggering point s_t, and the minimizer, IP_t^*, from (11) is the order-up-to level S_t. These conditions also imply that $TC_{III}(Q_0)$ in (P3.2) is well behaved in Q_0. This leads to the following theorem.

Theorem 5. Under conditions of Theorem 4 the optimal Q_0 in (P3.2) is unique.

Although, as illustrated in the preceding two sections, Model I and Model II can accommodate forecast revision in the preseason as well as additional updates based on observed demands under mild regularity conditions since, the cost function $h_\tau(x_\tau)$ conditional on the sufficient statistic is convex, we suspect that Model III, even under the conditions of Theorem 4, can not accommodate forecast revision since, the cost function $h_t(x_t)$ is unimodal, not necessarily convex.

2.4.5. Related Literature

While variants of Models I and II have appeared in the literature, we are not aware of any work that has established the optimality of (s, S) policies for the dynamically determined second order. Interestingly, Milner and Kouvelis (2002, 2005) advocated the use of this policy in the lost-sales version of this problem. They presented results with computational experience in finding heuristically determined values of s and S for the case when demand (rate) follows a Brownian motion. In contrast, under the conditions of Theorem 4, determining the optimal choices for Q_0 and the second order reduces to a series of one-variable optimization problems each with a unique solution.

2.5. Conclusions

In this chapter we have conducted a comprehensive analysis of the newsvendor problem with a second order opportunity by studying three model variations that differ in the timing of the second order. In all three models, the first order is placed for delivery at the beginning of the season. In Model I, the second order quantity is determined at the beginning of the season for delivery at a specified time. In Model II, the second order quantity is determined at some pre-specified time that can be any where during the season. In Model III, both the timing and quantity of the second order are determined dynamically.

Our focus has been on elucidating the structure of optimal ordering policies. For the static and partially dynamic cases, our integrative approach insightfully reveals the intuitive structure of the optimal inventory policy for this important problem setting. And, our approach also allows us to explain, why the policy structure may be significantly more complex under the most general fully dynamic case. In addition, we are able to show a correspondence between our modeling framework and many papers in the literature, thereby providing a unifying perspective on this class of problems.

The key to our analysis is casting all three models as sequential decision-making problems, allowing us to reduce the optimization problems into sequential and embedded searches for decision variables. While we have considered cases where demand is backordered until the second order is delivered, our modeling approach can support the analysis of variants of this problem in which demand may only be filled from inventory-on-hand, as in the models considered by Milner and Kouvelis (2002). Therefore, it is easy to understand why in their static and partially dynamic cases, base-stock policies are optimal. While we are unable to fully characterize the optimal structure for the fully dynamic case, we are able to identify conditions under which the optimal ordering policy is of the form (s,S), a theoretically appealing inventory control policy. We are also able to show that under more general conditions the optimal control policy may be more complex. It remains to be determined whether (s,S) policies can be optimal when demand may only be filled from inventory on-hand.

CHAPTER 3. STOCHASTIC MULTI-PERIOD MODEL WITH LIMITED ORDER OPPORTUNITIES

3.1. Introduction

This chapter considers a discrete-time finite-horizon stochastic inventory problem where the decision maker observes the inventory position at the beginning of every period and decides whether or not to place an order and in what quantities. The problem differs from the classical multi-period stochastic inventory problem because we assume an upper limit on the number of orders that can be placed over the problem horizon. If the problem horizon consists of T periods and N is the upper limit on the number of orders then we assume $N \leq T$. The decision maker gets N opportunities to place orders over T periods. It is in this sense that we call this a stochastic inventory problem with limited order opportunities.

Facing these limited order opportunities, the decision maker needs to optimally utilize them over T periods. A key feature of this decision problem is that, in addition to considering the inventory position, the decision maker will also need to consider the remaining order opportunities to decide when to place an order and how much. The focus is on the structure of the optimal policy.

3.1.1. Motivation

There are several motivations for us to study this problem. If $N = T$, then our problem becomes the same as the classical inventory problem where an order can be placed in every period. Since we permit $N \leq T$, our results add to the literature providing a generalization of the classical inventory problem.

Our model provides results for further analysis of the value of order flexibility (Milner and Kouvelis, 2002). Not only does this require the flexibility of determining order quantity dynamically, but also it requires the flexibility of determining dynamically when to place orders. In the third model of Milner and Kouvelis (2002), the timing and quantity of an order – the second order – need to be determined dynamically over a selling season. A conceptual extension of their problem is how to optimally place multiple orders over a selling season composed of many small time intervals, leading to a stochastic multi-period inventory problem with limited order opportunities.

Another motivation for the problem comes from the situation when a retailer has to share its ordering resources among two or more different items (say, Item 1 and Item 2). Consider a retailer who carries inventory of two different items that he buys from two different suppliers. Assume that the retailer has only one truck and replenishes inventory on a weekly basis. The retailer can replenish only one item in a week because he has only one truck. Thus, every week, the retailer will need to decide whether to replenish Item 1 or Item 2. This is a complex problem with no known solution to the best of our knowledge. While we do not provide a solution to this complex problem, our problem should serve as a sub problem in solving this and similar more complex problems.

3.1.2. Preview of Main Assumptions and Results

The specific problem that we consider assumes that the problem horizon has T periods. The on-hand inventory at the beginning of period 1 is given. Demand is independent across periods and has PF_2 density. The decision maker first updates the inventory position at the beginning of a period and then decides whether or not to place an order, and how much if ordering. The delivery lead time for an order is assumed to be fixed and known; it could be zero or some positive integer. The unit purchase price is assumed to be non-decreasing as we get closer to the end of the problem horizon. The order cost could include a fixed cost in addition to the linear purchase price. Demands not met in a period from on-hand inventory are backordered. Unmet demands at the end of the last period are assumed to be satisfied by an emergency order. End-of-period leftovers in every period are charged a linear holding cost. Similarly, the model charges a linear backorder cost for a period for backorders at the end of the period.

The focus of the chapter is on establishing the form of the optimal policy. The chapter contributes by showing that a time varying (s, S) policy is optimal. Under this policy, the decision maker places an order to raise the inventory position to S whenever it is less than s. In fact, s and S depend on the number of periods left in the problem horizon and on the number of the remaining orders.

3.1.3. Organization of this Chapter

The remainder of this chapter is organized as follows. In section 2, we discuss our results with respect to the previous literature. In section 3, we describe and analyze a one-order opportunity model. This model is extended in section 4 to allow for multiple order opportunities. In section 5, we end this paper with some conclusions and discussion of the future research.

3.2. Linkage to the Literature

Starting from Arrow, Harris and Marschak (1951), traditional multi-period, stochastic inventory decisions have focused on how much to purchase in each period. Literature discussing them is plentiful. Our intent here is to interpret our results and link these to the known results, not to give a detailed review of the literature. More examples and detailed review can be seen in Hadley and Whitin (1963), Porteus (1990) and Zipkin (2000).

Several researchers have discussed the (s,S) inventory control policy. When the purchase cost is a linear plus a fixed cost, the well-known base-stock policy is not optimal. Assuming demand densities belong to PFF family and unmet demand is lost, Karlin (1958) discussed the optimal policy for multiple periods. He also sought sufficient conditions for the (s,S) policy to be optimal. Scarf (1960) introduced the K-convexity of a function and showed optimality of the (s,S) policy. Veinott (1966) proved that (s,S) policy is optimal under new conditions which do not imply and are not implied by conditions in Scarf (1960). Porteus (1971) derived a generalized (s,S) policy assuming demand density functions belong to Pólya density family. As shown, if the ordering cost does have a fixed component independent of order size, a time-varying (s,S) policy is optimal for each period when each period has an order opportunity. This research shows that the scarcity of order opportunities leads to the same structure of the optimal policy as a fixed order cost for each period.

In a continuous-time single season problem with two order opportunities, Milner and Kouvelis (2002, 2005) proposed models such that the time and quantity for the second order are determined dynamically as sales goes on during the selling season. They considered a time varying policy and computed corresponding parameters approximately to minimize the cost.

3.3. Single Order Opportunity

3.3.1. Model Formulation

The problem horizon consists of T periods, starting at time 1 and ending at time $T+1$. The periods are numbered as periods $1,2,\cdots,T$. The demand in each period is positive, random and independent across periods, and its density function falls in PF_2. The initial inventory x, or the inventory level at time 1 before placing an order, is a result of prior replenishment decisions, and assumed to be given. The decision maker gets opportunity to place an order at the beginning of each of periods $1,2,\cdots,T$; however, he can place at most one order throughout T periods. For ease of exposition, we assume that, no matter when the order is placed, it is delivered instantly. (In fact, we can see in Appendix B.5 that the case with positive delivery leadtime is analogous to the case with zero delivery leadtime.)

The unit purchase cost at time $T-\tau+1$ (or the time that is τ periods away from the end of the horizon) is c_τ, where c_τ is nonincreasing in τ. (Explicit inclusion of a fixed ordering cost does not change the essence of the math, and a detailed discussion for it can be seen in Appendix B.4) Shortages are backordered and the backorder cost at the end of period $T-\tau+1$ is charged at a rate of p_τ per unit. To rule out motive for carrying over shortages from time T to time $T+1$, we assume $p_1 > c_1$. Leftovers at the end of period $T-\tau+1$ are charged a holding cost at a rate of h_τ per unit. Leftovers at the end of period T have a salvage value v per unit ($0 \le v \le c_T$). For simplicity of exposition, we assume $p_\tau = p$, $h_\tau = 0$ and $v = 0$ for $\tau \ge 1$. (Explicit inclusion of time-variant p_τ, h_τ and v does not change the essence of the analysis, and a detailed discussion for it can be seen in Appendix B.3) The objective of the decision maker is to minimize the expected cost while optimally utilizing the one order opportunity.

Let $V_0(\tau,y)$ represent the expected cost over the last τ periods if there are zero order opportunities left and we have y units of inventory at the beginning. Then, we have

$$V_0(\tau,y) = p\int_y^\infty (\xi - y) f_\tau(\xi)d\xi + \int_0^\infty V_0(\tau-1, y-\xi) f_\tau(\xi)d\xi$$

with boundary condition $V_0(0,y)\equiv 0$, where $f_\tau(\bullet)$ is the PDF of the demand, ξ, in period $T-\tau+1$. The first term of $V_0(\tau,y)$ is the expected backorder cost in period $T-\tau+1$; and the second term represents the expected cost in periods $T-\tau+2$ through T.

Define

$$O_1(\tau,y,IP)=c_\tau(IP-y)+V_0(\tau,IP) \quad \text{... (12)}$$

$O_1(\tau,y,IP)$ represents the cost for the last τ periods if $IP\geq y$ and an order is placed at time $T-\tau+1$ to raise the inventory level from y to IP.

Let $V_1(\tau,y)$ denote the minimum expected cost over the last τ periods if there is 1 order opportunity left and we have y units of inventory at the beginning. Let $O_1(\tau,y)$ denote the minimum expected cost if an order is placed at time $T-\tau+1$. (Note that the order placed at time $T-\tau+1$ is received instantly. Also, 0 order opportunities will be left for the remaining $\tau-1$ periods.)

Let $D_1(\tau,y)$ denote the minimum expected cost if no order is placed at time $T-\tau+1$. Then we have

$$D_1(\tau,y)=p\int_y^\infty(\xi-y)f_\tau(\xi)d\xi+\int_0^\infty V_1(\tau-1,y-\xi)f_\tau(\xi)d\xi \quad \text{... (13)}$$

The first term of $D_1(\tau,y)$ is the expected backorder cost in period $T-\tau+1$; and the second is the minimum expected cost of the last $\tau-1$ periods.

3.3.2. Linkage to Model III

Notice that Model III, where the second order needs to be optimally and dynamically placed over T periods, is related to this model. The difference is on the way the shortages after the placement of orders are treated. While the model in this chapter assumes that any demand that cannot be satisfied from the orders placed is backordered, charged backorder cost for the remaining periods and eventually satisfied at the end of the last period through an emergency order. Model III assumes that any demand that cannot be satisfied from the second order is lost no matter in which period it occurred and is charged a lost-sale cost.

3.3.3. Optimization Problems

At the beginning of period $T-\tau+1$, two decisions are to be made: whether to place an order now and how much if ordering. Depending on which one has a lower cost between $O_1(\tau,y)$ and $D_1(\tau,y)$, the costs for ordering and no ordering, respectively, we make the choice on whether to order or not. That is,

$$V_1(\tau,y)=\min\{O_1(\tau,y),D_1(\tau,y)\} \text{ for } \tau\le T \qquad \text{... (P4.1)}$$

It is optimal not to place an order if $D_1(\tau,y)\le O_1(\tau,y)$. If $O_1(\tau,y)<D_1(\tau,y)$, it is optimal to place an order. The optimal inventory level *IP* for order placement is determined through the following optimization problem

$$O_1(\tau,y)=\min_{IP\ge y}\{O_1(\tau,y,IP)=c_\tau(IP-y)+V_0(\tau,IP)\} \qquad \text{... (P4.2)}$$

3.3.4. Analysis and Results

In what follows, we analyze (P4.1) and (P4.2) to dynamically determine the ordering time and order quantity for the order. We show that a time-varying "(s,S) inventory policy" is optimal. In this policy, if the current inventory level falls below s and there is an order opportunity left, then an order is placed to raise the inventory level to S.

From the expression for $V_0(\tau, y)$, since the cost in each period is convex, it can be seen that $V_0(\tau, y)$ is convex in y with the limits $\lim_{y \to -\infty} V_0(\tau, y) = \infty$ and $\lim_{y \to \infty} V_0(\tau, y) = \infty$. Therefore $c_\tau y + V_0(\tau, y)$ is convex in y. Let $IP^*(\tau)$ be the unique value of y that minimizes $c_\tau y + V_0(\tau, y)$.

$O_1(\tau, y, IP)$ in (12), as a function of IP, is convex in IP and is minimized at $IP = IP^*(\tau)$. Therefore $O_1(\tau, y)$ in (P4.2) has a value of $c_\tau IP^*(\tau) + V_0(\tau, IP^*(\tau)) - c_\tau y$ for $y < IP^*(\tau)$ and has a value of $V_0(\tau, y)$ for $y \geq IP^*(\tau)$.

We are now ready to prove that it is optimal not to place an order if $y \geq IP^*(\tau)$ at time $T - \tau + 1$.

Lemma 3. Assume we are at time $T - \tau + 1$ and the inventory level is y. If $y \geq IP^*(\tau)$, then it is optimal not to place an order, i.e., $D_1(\tau, y) \leq O_1(\tau, y)$.

Proof Cost for the last τ periods if we do not place an order is $D_1(\tau, y)$. Cost if we place an order is $V_0(\tau, y)$ by (P4.2) since $O_1(\tau, y, IP)$ is increasing in IP over $[IP^*(\tau), \infty)$. The proof follows since the $D_1(\tau, y)$ solution, compared to the $V_0(\tau, y)$, has a flexibility of placing an order for the last $\tau - 1$ periods. ■

By taking advantage of Lemma 3, (P4.1) becomes

$$V_1(\tau, y) = \begin{cases} \min\{O_1(\tau, y), D_1(\tau, y)\} & \text{if } y < IP^*(\tau) \\ D_1(\tau, y) & \text{if } y \geq IP^*(\tau) \end{cases} \qquad \text{... (14)}$$

We next focus on the case when $y < IP^*(\tau)$. To compare $O_1(\tau, y)$ and $D_1(\tau, y)$, it is sufficient to compare $\Theta(\tau)$ and $\text{Đ}_1(\tau, y)$, where $\Theta(\tau)$ denotes $c_\tau IP^*(\tau) + V_0(\tau, IP^*(\tau))$ and $\text{Đ}_1(\tau, y)$ denotes $D_1(\tau, y) + c_\tau y$. Then it is optimal to place an order if $\Theta(\tau) < \text{Đ}_1(\tau, y)$.

The question that needs to be answered next is: Given that we are at time $T - \tau + 1$ and inventory level $y < IP^*(\tau)$, for what values of y is it optimal to place an order? We answer this question in a recursive manner by starting from $\tau = 1$.

Case $\tau = 1$

In this case, we have an order opportunity at time T. With a beginning inventory level of y before the purchase decision, the cost if we place an order is $\Theta(1) - c_1 y$, and the cost if we do not place an order is $D_1(1, y)$. To decide whether to place an order, we compare $\Theta(1)$ and $\text{Đ}_1(1, y)$, knowing that $\Theta(1) = c_1 IP^*(1) + V_0(1, IP^*(1))$ and $\text{Đ}_1(1, y) = D_1(1, y) + c_1 y$. It is easy to see that $D_1(1, y) = V_0(1, y)$, thus $\text{Đ}_1(1, y)$ is convex in y with minimum attained at $IP^*(1)$, leading to $\Theta(1) \leq \text{Đ}_1(1, y)$ for $y < IP^*(1)$. To conclude, a base-stock policy is optimal for $\tau = 1$: If $y < IP^*(1)$, then it is optimal to place an order and raise the inventory level to $IP^*(1)$.

Case $\tau \geq 2$

We show that an (s, S) policy with $S = IP^*(\tau)$ is optimal for this case. We do so by showing that there exists $y = y^*(\tau) \leq IP^*(\tau)$ such that $\text{Đ}_1(\tau, y) = \Theta(\tau)$ at

$y^*(\tau)$, and that $Đ_1(\tau, y)$ stays below $\Theta(\tau)$ for $y \in (y^*(\tau), IP^*(\tau))$ and stays above $\Theta(\tau)$ for $y \in (-\infty, y^*(\tau))$. The existence of $y^*(\tau)$ where $Đ_1(\tau, y)$ and $\Theta(\tau)$ intersect follows from the two lemmas below.

Lemma 4. 1) $\lim_{y \to -\infty} Đ_1(\tau, y) = \infty$; 2) $\lim_{y \to \infty} Đ_1'(\tau, y) = c_\tau$.

Proof 1) Note that $Đ_1(\tau, y) = c_\tau y + D_1(\tau, y)$. As y reduces, $c_\tau y$ will reduce at a rate of c_τ. However the cost of purchase will not decrease in future periods (from model assumption that c_τ is non-increasing in τ) and there exists a positive backorder cost. Therefore, $Đ_1(\tau, y)$ will approach ∞ as y approaches $-\infty$.

2) As y increases, $c_\tau y$ will increase at a rate of c_τ. And no other cost will be charged when y approaches ∞. Therefore the marginal cost of $Đ_1(\tau, y)$ is c_τ as y approaches ∞. ■

Lemma 5. $Đ_1(\tau, IP^*(\tau)) \le \Theta(\tau)$.

Proof It is equivalent to showing that $D_1(\tau, IP^*(\tau)) \le V_0(\tau, IP^*(\tau))$ by their definitions. The inequality holds because the $D_1(\tau, IP^*(\tau))$ solution has flexibility of placing an order for the remaining periods while the $V_0(\tau, IP^*(\tau))$ solution does not have any remaining order opportunity. ■

To show that $Đ_1(\tau, y)$ stays below $\Theta(\tau)$ for $y \in (y^*(\tau), IP^*(\tau))$, it is sufficient to show that $Đ_1(\tau, y)$ is unimodal in y. We show it in the following theorem.

Theorem 6. For $\tau \geq 2$, we have

1) $D_1(\tau, y)$ is unimodal.
2) There exists a unique $y^*(\tau) < IP^*(\tau)$ such that

$$V_1(\tau,y) = \begin{cases} O_1(\tau,y) & \text{for } y < y^*(\tau) \\ D_1(\tau,y) & \text{for } y \geq y^*(\tau). \end{cases}$$

Therefore, $V_1(\tau,y)$ is unimodal in y.

Proof See Appendix B.1. ■

By Theorem 6, we actually show that the optimal policy at time $T-\tau+1$ is an (s, S) policy with parameters $s = y^*(\tau), S = IP^*(\tau)$. To summarize, we examined the question of how to dynamically make an order decision with respect to ordering time and quantity. We showed that a time-varying (s, S) policy is optimal.

3.4. Multiple Order Opportunities

In this section, we extend our model in section 3 to allow for multiple order opportunities. Specifically, we suppose the decision maker can place at most N orders over the last T periods.

3.4.1. Model Formulation

Suppose we are at time $T-\tau+1$. Let $V_k(\tau, y)$ denote the minimum expected cost over the last τ periods if there are k order opportunities left and we have y units of inventory. Let $O_k(\tau, y)$ denote the minimum expected cost if an order is placed at time $T-\tau+1$.

Define

$$O_k(\tau,y,IP)=c_\tau(IP-y)+p\int_{IP}^{\infty}(\xi-IP)f_\tau(\xi)d\xi+\int_0^{\infty}V_{k-1}(\tau-1,IP-\xi)f_\tau(\xi)d\xi \quad \text{... (15)}$$

for $k \geq 1$ and any y. Then $O_k(\tau,y,IP)$ represents the minimum expected cost for the last τ periods if $IP \geq y$ and an order is placed at time $T-\tau+1$ to raise the inventory level from y to IP. The first term of $O_k(\tau,y,IP)$ is the purchase cost for $IP-y$ units. The second term is the expected backorder cost in period $T-\tau+1$. And the last term is the expected minimum cost for the last $\tau-1$ periods.

Let $D_k(\tau,y)$ denote the minimum expected cost if no order is placed at time $T-\tau+1$. Then, we have

$$D_k(\tau,y)=p\int_{y}^{\infty}(\xi-y)f_\tau(\xi)d\xi+\int_0^{\infty}V_k(\tau-1,IP-\xi)f_\tau(\xi)d\xi \quad \text{... (16)}$$

for $k \geq 1$ and any y. The first term of $D_k(\tau,y)$ is the expected backorder cost in period $T-\tau+1$. And the second term is the minimum expected cost over last $\tau-1$ periods.

3.4.2. Optimization Problems

The decision maker makes the optimal choice on whether to order by solving the following optimization problem:

$$V_k(\tau,y)=\min\{O_k(\tau,y),D_k(\tau,y)\} \quad \text{... (P4.3)}$$

It is optimal not to place an order if $D_k(\tau,y) \leq O_k(\tau,y)$. If $O_k(\tau,y) < D_k(\tau,y)$, then it is optimal to place an order. The optimal inventory level after the order placement is determined by the following optimization problem

$$O_k(\tau, y) = \min_{IP \geq y} \{ O_k(\tau, y, IP) \} \qquad \text{... (P4.4)}$$

3.4.3. Analysis and Results

From (15) and (16), it is easy to see that $O_k(\tau, y, IP)$ and $D_{k\text{-}1}(\tau, IP)$ have the following relationship

$$O_k(\tau, y, IP) = c_\tau IP + D_{k-1}(\tau, IP) - c_\tau y \qquad \text{... (17)}$$

In view of the expression above, we need to analyze the function defined below to study $O_k(\tau, y, IP)$

$$\bar{D}_k(\tau, y) = c_\tau y + D_k(\tau, y) \qquad \text{... (18)}$$

In particular, we will prove by induction that $\bar{D}_k(\tau, y)$ is unimodal in y. Recall it was proved that $\bar{D}_1(\tau, y)$ is unimodal in y for any τ in section 3. We proceed with the assumption that $\bar{D}_{k-1}(\tau, y)$ is unimodal in y for any τ, therefore $O_k(\tau, y, IP)$, as a function of IP, is unimodal in IP.

Denote $IP_k^*(\tau)$ as the point that minimizes $O_k(\tau, 0, IP)$ with respect to IP. That is,

$$IP_k^*(\tau) = \min_{IP} \{ O_k(\tau, 0, IP) \}.$$

By the unimodality of $O_k(\tau, 0, IP)$ with respect to IP, we can see that $IP_k^*(\tau)$ is actually the order-up-to level if an order is placed at time $T - \tau + 1$, and the corresponding minimum expected cost, $O_k(\tau, y)$ in (P4.4), is

$$O_k(\tau, y) = c_\tau IP_k^*(\tau) + D_{k\text{-}1}\left(\tau, IP_k^*(\tau)\right) - c_\tau y \ .$$

We are now ready to prove that if $y \geq IP_k^*(\tau)$, it is optimal not to place an order.

Lemma 6. If $y \geq IP_k^*(\tau)$, then it is optimal not to place an order.

Proof Cost for the last τ periods if we do not place an order is $D_k(\tau, y)$. Cost if we place an order is $O_k(\tau, y, y)$ (or $D_{k-1}(\tau, y)$) since $O_k(\tau, y, IP)$ is non-decreasing in IP over $\left[IP_k^*(\tau), \infty\right)$. The proof follows since the $D_k(\tau, y)$ solution has one more order opportunity for the remaining $\tau - 1$ periods than the $D_{k-1}(\tau, y)$ solution has.■

By taking advantage of Lemma 6, (P4.3) becomes

$$V_k(\tau, y) = \begin{cases} \min\left\{O_k(\tau, y), D_k(\tau, y)\ \right\} & \text{if } y < IP_k^*(\tau) \\ D_k(\tau, y) & \text{if } y \geq IP_k^*(\tau) \end{cases}.$$

We next focus on the case when $y < IP_k^*(\tau)$ to compare $O_k(\tau, y)$ and $D_k(\tau, y)$. By (17), to do so, it is sufficient to compare $\Theta_k(\tau)$ and $\text{Đ}_k(\tau, y)$, where $\Theta_k(\tau)$ denotes $c_\tau IP_k^*(\tau) + D_{k-1}\left(\tau, IP_k^*(\tau)\right)$.

The question that needs to be answered next is: Given that we are at time $T - \tau + 1$ and have a inventory level $y < IP_k^*(\tau)$, for what values of y is it optimal to place an order? We answer this question by proving that $\text{Đ}_k(\tau, y)$ is unimodal in y for any τ, analogous to the single order opportunity case in section 3.

Theorem 7.

1) For $\tau \le k$, $D_k(\tau, y)$ is convex.

2) For $\tau \ge k+1$, $D_k(\tau, y)$ is unimodal. Therefore, there exists a unique $y_k^*(\tau) < IP_k^*(\tau)$ such that

$$V_k(\tau,y) = \begin{cases} O_k(\tau,y) & \text{for } y < y_k^*(\tau) \\ D_k(\tau,y) & \text{for } y \ge y_k^*(\tau) \end{cases}$$

Therefore, an (s,S) policy with parameters $s = y_k^*(\tau), S = IP_k^*(\tau)$ is optimal, and $V_k(\tau,y)$ is unimodal in y.

Proof See Appendix B.2. ■

3.5. Conclusions and Ideas for Future Research

To conclude, we have examined the question of how to dynamically place orders in a finite horizon setting where the number of orders that can be placed is less than the number of periods that can be used for order placing. Under our assumptions, we have shown that a time varying (s,S) type policy is optimal.

Note that our focus is on the structure of the optimal policy, the corresponding computational issues are not addressed. Some researchers have discussed the computational issues for (s,S) policies (see Veinott and Wagner, 1965, Federgruen and Zipkin, 1984, Zheng and Federgruen, 1991). It can be seen that the scarcity of order opportunities leads to the same structure of optimal policy as for fixed order costs. It might be interesting to explore the feasibility of applying the known algorithms, or to develop new algorithms, to calculate the optimal policy parameters. Heuristics and bounds for the optimal policies are also interesting research topics.

CHAPTER 4. STOCHASTIC MULTI-PERIOD MODELS WITH TWO SUPPLY MODES

4.1. Introduction

4.1.1. Motivation

Outsourcing/In-Sourcing is an important consideration for retailers and manufacturers in today's global supply chain environment. Facing domestic issues like higher cost for labor, health care, raw material, and energy, etc., more and more companies are finding it economical to shift their production (or purchase) to countries abroad where costs are lower. However, delivery leadtimes for outsourcing from abroad could be long due to factors such as long shipping distances, queuing time at transfer stations (like ports, customs, etc.), security inspections, etc. Companies are using faster delivery methods to counter the inflexibility associated with long leadtimes. Faster-delivery methods include faster delivery mode and/or local production, etc. These methods provide the company a capability to react quickly to uncertain demands.

A natural decision question when considering outsourcing is how to balance the order quantities between suppliers with different delivery times. We build a stochastic, periodic-review inventory model to reflect this decision across periods with two suppliers – a fast supplier and a slow supplier. In this model, the general question for the decision maker is how to make optimal inventory replenishment decisions in each period (or, equivalently, determine optimal order quantities for the fast order and the slow order). The criterion is to minimize the expected cost over the problem horizon.

4.1.2. Preview of Main Assumptions and Results

The decision maker gets opportunity to place two orders at the beginning of each period: one order has a slow delivery time and the other one has a fast delivery time. The delivery time for the fast order is negligible. The delivery time for the slow order is a fixed value less than the review period. Demand is independent across periods and has PF_2 density. Shortages are either backordered or lost. Leftovers incur a holding cost. Shortage and holding cost are linearly charged. The rates for these costs and demand in each period may vary over time. There is no fixed cost associated with ordering. The unit purchase price from the fast supplier could be higher than that from the slow supplier. There may or may not be information updating on demand distribution before the order decisions.

The chapter contributes by showing that there are unique "order up to" levels to determine the order quantities from these two suppliers. We identify conditions when it is optimal to order from just one supplier or from both. In case it is optimal to order from both in a period, we show that at the beginning of the period, if the beginning inventory level is between a certain pair of points, then it is optimal to raise the inventory position to the higher point through a slow order. However, if the beginning inventory position is lower than the lower point, then the inventory level is first raised up to this point through a fast order and then the inventory position is raised up to the higher point through a slow order. If the beginning inventory is higher than the higher point, no order needs to be placed. The optimal policies in this chapter are supported by the property that the cost is unimodal in the beginning inventory position and convex in the beginning inventory level. We need the PF_2 density assumption to prove this property.

4.1.3. Organization of this Chapter

The remainder of this chapter is organized as follows. In Section 2, we review the related literature. In Section 3, we describe and analyze the finite horizon model. Some managerial insights are discussed in Section 4. We end this chapter with our conclusions in Section 5.

4.2. Linkage to the Literature

The related literature includes papers studying the optimal inventory replenishment decisions for two delivery modes. One way to interpret two delivery modes is through "fast order" and "slow order". The delivery with shorter time is called a *fast order* and the one with longer time is called a *slow order*. When the beginning on-hand inventory is too low, the probability that shortages will occur in a period is high and the shortage cost could be high. Therefore, it may be economical to have a fast order to replenish the on-hand stock. In the models of Barankin (1961) and Daniel (1963), the fast order is delivered instantly and the slow order is delivered at the end of the review period when it is placed. While Barankin derived the one-period optimal policy assuming the fast order is of a fixed amount, Daniel derived the multi-period optimal policy assuming the fast order can be any amount limited from above. The model that we present differs from Barankine and Daniel in that we allow the delivery leadtime for the slow order to be less than a period.

Some researchers have studied a periodic inventory system with two supply modes where the regular review cycle is greater than the supply leadtime and shortages are backordered (Chiang and Gutierrez, 1998, Fox et al., 2005, Fukuda, 1964 and Yi and Scheller-Wolf, 2003). In the model by Chiang and Gutierrez (1998), a review cycle consists of several periods. The delivery time for a fast order is one period and the delivery time for a slow order is τ periods with τ greater than 2. A slow order is placed only in the first period of a review cycle and a fast order can be placed in each period. In the model of Fox et al. (2005), the review cycle is 1 period and delivery leadtimes for two suppliers are zero. The supply with lower unit cost has a fixed cost and the supply with higher unit cost has no fixed cost. Still assuming zero leadtime for two suppliers, Yi and Scheller-Wolf (2003) differentiate supply by price uncertainty: one supplier has a constant unit price with limits on purchase quantity; the other one has a fixed cost plus fluctuating unit price with no limits on purchase quantity. The model with three supply modes in Fukuda (1964) assumes that a review cycle consists of 2 periods. Order decisions are made only in the first period of a cycle. When the quantity with the longest delivery is zero, that model becomes a variant of our model. Our model differs from all these models in how we model the backorder cost. We allow the backorder cost to depend on the number of periods between when the backorder originated and when the backordered demand got met. In our model, there could be penalty (in addition to regular backorder cost) if a backorder is carried from one interval to next. This penalty could be positive or negative. We call this feature "penalty feature" of the model.

Another way to interpret two delivery modes is through "reserved order" and "normal order". The delivery with longer time is called a *reserved order*, which is used to raise the inventory level in a future period. The delivery with shorter time is called a *normal order*, which is used to raise the inventory level in the current period. When the beginning inventory is not too low, although it may not be economical to have normal order delivered right now, we may order for future with a lower cost per unit. Some researchers have shown that it is economical to make use of reserved orders (Bulinskaya, 1964a, 1964b, Feng et al., 2005, Fukuda, 1964, Neuts,1964, Sethi et al., 2001 and Whittemore and Saunders, 1977). In their models, shortages in a period incur same per-unit shortage cost, no matter whether it is met by the reserved delivery at the end of the period or by delivery at the beginning of the next period. Therefore the delivery at the end of a period is to only raise the on-hand inventory level at a cheaper price.

4.3. Model and Analysis

4.3.1. Model Formulation

The problem horizon consists of T periods, starting at time 1 and ending at time $T+1$. The periods are numbered $1,2,\cdots,T$. With a beginning inventory level of $\underline{z}$ in period $T-t+1$, $z-\underline{z}$ units are purchased for the fast order and $IP-z$ units are purchased for the slow order. The fast order is delivered instantly and the slow order takes $\lambda(0<\lambda<1)$ periods for its delivery so that each period is divided into two intervals: the first interval is the part before the delivery of the slow order; and the second interval is the remaining part of the review period. The unit purchase cost is $c_{t,e}$ for the fast order and is $c_{t,n}$ for the slow order. The demand in each period and each interval is random, independent across periods and intervals and has PF_2 density. The cumulative demand in the first interval of period $T-t+1$ in excess of z but not beyond IP is backordered at a rate of p_{t,λ^-} per unit; and those in excess of IP (or not satisfied in period t) are backordered at a rate of p_t per unit. The unmet demand in the second interval is backordered at a rate of p_{t,λ^+} per unit. Consistent with the "penalty feature" of our model, p_t can be different from the sum of p_{t,λ^-} and p_{t,λ^+}. The on-hand inventory right before the delivery of the slow order (or at the end of the first interval) in period $T-t+1$ is charged a holding cost at a rate of h_{t,λ^-} per unit; and the inventory at the end of the second interval is charged a holding cost at a rate of h_{t,λ^+} per unit. The objective of the decision maker is to minimize the expected cost over T periods while choosing the quantities for the fast and slow orders in each period. This cost is the sum of the purchase cost of the two orders, the cost for backordering demand and the cost of holding inventory.

Define

$$L_t(z,IP) = p_{t,\lambda^-} E\left[\min\left\{(\xi_{\lambda^-} - z)^+, (IP - z)\right\}\right] + h_{t,\lambda^-} E\left[(z - \xi_{\lambda^-})^+\right]$$

where ξ_{λ^-} is demand in the first interval of period $T-t+1$ with PDF $\tilde{f}_{t,\lambda^-}(\cdot)$. Then, for $z \le IP$, $L_t(z,IP)$ is the sum of the expected backorder cost for the demand met from the delivery of the slow order and the expected holding cost at the end of the first interval of period $T-t+1$, with a beginning inventory level z and an on-transit inventory $IP-z$. The first term of $L_t(z,IP)$ is the expected cost for backordering the demand in the first interval which is not satisfied from the on-hand inventory z but satisfied from the slow order; and the second term is the expected holding cost for the first interval. When the beginning inventory z is less than zero, at most IP units of demand can be backordered and no holding cost is charged. When z is greater than zero, only demands greater than z and less than or equal to IP can be backordered and satisfied from the slow order, and the expected holding cost is $h_{t,\lambda^-} E\left[(z - \xi_{\lambda^-})^+\right]$.

Define

$$R_t(y) = \begin{cases} p_{t,\lambda^+} E\left[(\xi_{\lambda^+} - y)^+\right] + h_{t,\lambda^+} E\left[(y - \xi_{\lambda^+})^+\right] + E\left[V_{t-1}(y - \xi_{\lambda^+})\right] & y \ge 0 \\ p_t (y)^- + p_{t,\lambda^+} \tilde{\mu}_{t,\lambda^+} + E\left[V_{t-1}(y - \xi_{\lambda^+})\right] & y < 0 \end{cases}$$

for any y, where y is the beginning inventory level for the second interval of period $T-t+1$ (after receiving the slow order) and ξ_{λ^+} is its demand. $V_{t-1}(\underline{z})$ is the minimum expected cost for the last $t-1$ periods, with the beginning inventory level of $\underline{z}$. Then $R_t(y)$ is the minimum expected cost incurred in the second interval of period $T-t+1$ and the last $t-1$ periods, with a beginning inventory level of y. For $y \geq 0$, the first term of $R_t(y)$ is the expected cost of backordering the demands which are not satisfied in period $T-t+1$; the second term is the expected holding cost incurred in the second interval; and, the last term is the minimum expected cost for the last $t-1$ periods. For $y<0$, the first term of $R_t(y)$ is the expected backorder cost for the demand in the first interval unmet from the delivery of the slow order; the second term is the expected backorder cost for the demand in the second interval; and, the last term is the minimum expected cost for the last $t-1$ periods.

The cost function for the last t periods can be given recursively as follows. Define

$$V_t(\underline{z},z,IP) = c_{t,e}(z-\underline{z}) + c_{t,n}(IP-z) + L_t(z,IP) + \int_0^\infty R_t(IP-\xi_{\lambda^-})\tilde{f}_{t,\lambda^-}(\xi_{\lambda^-})d\xi_{\lambda^-}$$

Then, for $\underline{z} \leq z \leq IP$, $V_t(\underline{z},z,IP)$ is the minimum expected cost for the last t periods, with a beginning inventory level of $\underline{z}$, a purchase of $z-\underline{z}$ units using the fast order and a purchase of $IP-z$ units using the slow order. $L_t(z,IP)$ is the expected cost incurred in the first interval of period $T-t+1$, with a beginning inventory level z and an in-transit inventory of $IP-z$. $R_t(y)$ is the minimum expected cost incurred in the second interval of period $T-t+1$ and the last $t-1$ periods, with a beginning inventory level of y.

4.3.2. Model Assumptions

We make the following assumptions: 1) $c_{t,n} + p_{t,\lambda^-} > c_{t,e}$: it saves to meet a demand through the fast order than first backorder it and then satisfy it through the slow order; 2) $c_{t,e} - c_{t,n} + h_{t,\lambda^-} > 0$: it saves to purchase a unit through the slow order than to first buy it through the fast order and then hold it till the second interval. In our description above, we also assume that λ is constant for each period. It will be seen from our analysis later that λ can vary across periods as long as it is between 0 and 1. We assume all unmet demand are backordered, $p_t > p_{t,\lambda^-}$ and $p_t > p_{t,\lambda^+}$. (Discussion for the shortage assumption can be seen in Appendix C.7.) Finally, we also assume that $c_{t,n} + p_{t,\lambda^-} - p_t < c_{t-1,e}$ for $2 \le t \le T$ and $c_{1,n} + p_{1,\lambda^-} - p_1 < 0$: rather than meeting a unit of demand in the first interval through the fast order in the next period, it saves to meet it in the current period through the slow order.

4.3.3. Optimization Problem

We are now ready to present the decision problem in period $T-t+1$ below.

$$V_t(\underline{z}) = \min_{z \ge \underline{z}, IP \ge z} \{V_t(\underline{z}, z, IP)\} \qquad \text{... (P5.1)}$$

That is, the decision in period $T-t+1$ is to choose the optimal inventory level z_t^* and the optimal inventory position IP_t^* (or, equivalently, the fast order quantity $z_t^* - \underline{z}$ and the slow order quantity $IP_t^* - z_t^*$). $V_t(\underline{z})$ is therefore the minimum expected cost for the last t periods with a beginning inventory level of $\underline{z}$ before the order decision. We assume $V_0(\underline{z}) \equiv 0$.

4.3.4. Analysis

We solve (P5.1) in a recursive way. In particular, we show that: 1) $V_t(\underline{z})$ is convex and $V_t'(\underline{z}) = -c_{t,e}$ for $\underline{z} \le 0$, provided that $V_{t-1}(\underline{z})$ is convex and $V_{t-1}'(\underline{z}) = -c_{t-1,e}$ for $\underline{z} \le 0$; and 2) $V_1(\underline{z})$ is convex and $V_1'(\underline{z}) = -c_{1,e}$ for $\underline{z} \le 0$.

Since $V_t(\underline{z})$ is determined through (P5.1) and $V_t(\underline{z}, z, IP)$ is determined by $R_t(y)$ and $L_t(z, IP)$, we first study the properties of functions $R_t(y)$ and $L_t(z, IP)$. Recall that

$$R_t(y) = \begin{cases} p_{t,\lambda^+} E\left[\left(\xi_{\lambda^+} - y\right)^+\right] + h_{t,\lambda^+} E\left[\left(y - \xi_{\lambda^+}\right)^+\right] + E\left[V_{t-1}\left(y - \xi_{\lambda^+}\right)\right] & y \geq 0 \\ p_t(y)^- + p_{t,\lambda^+}\tilde{\mu}_{t,\lambda^+} + E\left[V_{t-1}\left(y - \xi_{\lambda^+}\right)\right] & y < 0 \end{cases}$$

$$= \begin{cases} \left(p_{t,\lambda^+} + h_{t,\lambda^+}\right) \int_y^\infty \left(\xi_{\lambda^+} - y\right) \tilde{f}_{t,\lambda^+}\left(\xi_{\lambda^+}\right) d\xi_{\lambda^+} - h_{t,\lambda^+}\left(\tilde{\mu}_{t,\lambda^+} - y\right) \\ \quad + \int_0^\infty V_{t-1}\left(y - \xi_{\lambda^+}\right) \tilde{f}_{t,\lambda^+}\left(\xi_{\lambda^+}\right) d\xi_{\lambda^+} & y \geq 0 \\ -p_t y + p_{t,\lambda^+}\tilde{\mu}_{t,\lambda^+} + \int_0^\infty V_{t-1}\left(y - \xi_{\lambda^+}\right) \tilde{f}_{t,\lambda^+}\left(\xi_{\lambda^+}\right) d\xi_{\lambda^+} & y < 0 \end{cases}$$

where $\tilde{f}_{t,\lambda^+}\left(\xi_{\lambda^+}\right)$ and $\tilde{\mu}_{t,\lambda^+}$, respectively, are the PDF and the mean for the demand in the second interval of period $T - t + 1$. Based on the expression for $R_t(y)$ above, we can show that $R_t(y)$ is convex in y. In fact, taking derivatives for $R_t(y)$ yields

$$\frac{dR_t(y)}{dy} = \begin{cases} -\left(p_{t,\lambda^+} + h_{t,\lambda^+}\right) \int_y^\infty \tilde{f}_{t,\lambda^+}\left(\xi_{\lambda^+}\right) d\xi_{\lambda^+} + \int_0^\infty V_{t-1}'\left(y - \xi_{\lambda^+}\right) \tilde{f}_{t,\lambda^+}\left(\xi_{\lambda^+}\right) d\xi_{\lambda^+} + h_{t,\lambda^+} & y \geq 0 \\ -p_t + \int_0^\infty V_{t-1}'\left(y - \xi_{\lambda^+}\right) \tilde{f}_{t,\lambda^+}\left(\xi_{\lambda^+}\right) d\xi_{\lambda^+} & y < 0 \end{cases}$$

$$\frac{d^2R_t(y)}{dy^2} = \begin{cases} \left(p_{t,\lambda^+} + h_{t,\lambda^+}\right) \tilde{f}_{t,\lambda^+}(y) + \int_0^\infty V_{t-1}''\left(y - \xi_{\lambda^+}\right) \tilde{f}_{t,\lambda^+}\left(\xi_{\lambda^+}\right) d\xi_{\lambda^+} & y \geq 0 \\ \int_0^\infty V_{t-1}''\left(y - \xi_{\lambda^+}\right) \tilde{f}_{t,\lambda^+}\left(\xi_{\lambda^+}\right) d\xi_{\lambda^+} & y < 0 \end{cases}$$

The convexity of $R_t(y)$ follows since $\frac{d^2R_t(y)}{dy^2} \geq 0$.

Similarly, we can study the properties of $L_t(z, IP)$. With a little algebra, $L_t(z, IP)$ can be expressed as

$$L_t(z, IP) = L1_t(z) - L2_t(IP) \qquad \text{where}$$
$$L1_t(z) = p_{t,\lambda^-} \int_z^{\infty} (\xi_{\lambda^-} - z) \tilde{f}_{t,\lambda^-}(\xi_{\lambda^-}) d\xi_{\lambda^-} + h_{t,\lambda^-} \int_0^z (z - \xi_{\lambda^-}) \tilde{f}_{t,\lambda^-}(\xi_{\lambda^-}) d\xi_{\lambda^-}$$
$$L2_t(IP) = p_{t,\lambda^-} \int_{IP}^{\infty} (\xi_{\lambda^-} - IP) \tilde{f}_{t,\lambda^-}(\xi_{\lambda^-}) d\xi_{\lambda^-}$$

Based on the above expression for $L_t(z, IP)$, it is easy to derive the following conclusions for $L_t(z, IP)$: 1) $L_t(z, IP)$ is separable in z and IP; 2) $L_t(z, IP)$ is convex in z since $L1_t(z)$ is the newsvendor-type expected shortage and holding cost for a beginning inventory z; and 3) $L_t(z, IP)$ is concave in IP since $L2_t(IP)$, being newsvendor-type expected shortage cost as if IP were the beginning inventory, is convex.

In the above analysis, if there is Bayesian updating on the demand distributions by using conjugate family, then $\tilde{f}_{t,\lambda^-}$ and $\tilde{f}_{t,\lambda^+}$ represent the updated distributions, and they still belong to PF_2 family. In this sense, Bayesian updating does not change our results.

We are now ready to study the properties of $V_t(\underline{z}, z, IP)$. In particular, we show that $V_t(\underline{z}, z, IP)$ is unimodal in IP and is convex in z. We demonstrate it in the following theorem. (All the proofs for Theorems and Results below are provided in the Appendix C.1-C.6)

Theorem 8.

1) $V_t(\underline{z}, z, IP)$ is unimodal in IP, and there is a unique $IP_t^*(>0)$ satisfying

$$\frac{\partial V_t(\underline{z}, z, IP)}{\partial IP} = c_{t,n} + p_{t,\lambda^-} \int_{IP}^{\infty} \tilde{f}_{t,\lambda^-}(\xi_{\lambda^-}) d\xi_{\lambda^-} + \int_0^{\infty} \frac{\partial R_t(IP - \xi_\lambda)}{\partial IP} \tilde{f}_{t,\lambda^-}(\xi_{\lambda^-}) d\xi_{\lambda^-} = 0.$$

2) $V_t(\underline{z}, z, IP)$ is convex in z, and there is a unique $z_t^*(>0)$ satisfying

$$\frac{\partial V_t(\underline{z},z,IP)}{\partial z}=\left(c_{t,e}-c_{t,n}+h_{t,\lambda^-}\right)-\left(p_{t,\lambda^-}+h_{t,\lambda^-}\right)\int_z^\infty \tilde{f}_{t,\lambda^-}\left(\xi_{\lambda^-}\right)d\xi_{\lambda^-}=0$$

From Theorem 8 the optimal ordering rule at time $T-t+1$ is: 1) to purchase $IP_t^*-z_t^*$ units for the slow order and purchase $z_t^*-\underline{z}$ units for the fast order if $\underline{z}\le z_t^*$; 2) to purchase $IP_t^*-\underline{z}$ units for the slow order if $z_t^*<\underline{z}\le IP_t^*$ and not to purchase if $\underline{z}>IP_t^*$. Then by the definition of $V_t(\underline{z})$, we have

$$V_t(\underline{z})=\begin{cases}V_t\left(\underline{z},z_t^*,IP_t^*\right) & \text{If } \underline{z}\le z_t^* \\ V_t\left(\underline{z},\underline{z},IP_t^*\right) & \text{If } z_t^*<\underline{z}\le IP_t^* \\ V_t\left(\underline{z},\underline{z},\underline{z}\right) & \text{If } \underline{z}>IP_t^*\end{cases}.$$

We next show that $V_t(\underline{z})$ is convex in $\underline{z}$. We show it in the following theorem.

Theorem 9. $V_t(\underline{z})$ is convex in $\underline{z}$, and $V_t'(\underline{z})=-c_{t,e}$ for $\underline{z}\le 0$.

In the above analysis, we implicitly assumed that $IP_t^*\ge z_t^*$. If this is not so then only the order for the fast delivery is placed. And the target level IP_t^* satisfies

$$c_{t,e}+\left.\frac{\partial L_t(z,IP)}{\partial z}\right|_{z=IP}+\left.\frac{\partial L_t(z,IP)}{\partial IP}\right|_{z=IP}+\int_0^\infty \frac{\partial R_t\left(IP-\xi_{\lambda^-}\right)}{\partial IP}\tilde{f}_{t,\lambda^-}\left(\xi_{\lambda^-}\right)d\xi_{\lambda^-}=0.$$

In this case, the optimal ordering rule is: to purchase $IP_t^*-\underline{z}$ units for the fast order if $\underline{z}\le IP_t^*$ and not to purchase if $\underline{z}>IP_t^*$. It can be seen that $V_t(\underline{z})$ is still convex in $\underline{z}$.

To complete the induction, let us study the properties of $V_1(\underline{z})$ as the induction base. We form it in the following theorem.

Theorem 10.

1) $V_1(\underline{z}, z, IP)$ is unimodal in IP.

2) $V_1(\underline{z}, z, IP)$ is convex in z.

3) $V_1(\underline{z})$ is convex in $\underline{z}$, and $V_1^{'}(\underline{z}) = -c_{1,e}$ for $\underline{z} \le 0$.

By Theorem 10, we see that $V_1(\underline{z})$ is convex in $\underline{z}$ and that $V_1^{'}(\underline{z}) = -c_{1,e}$. This is the induction base. This completes our recursive way to solve (P5.1).

4.4. Further Discussion

Based on the results in Section 3, several managerial insights on how to balance the order quantities between the fast order and the slow order (i.e. between z_t^* and $IP - z_t^*$ for given IP) can be derived. In particular, we examine: 1) How do the cost parameters affect z_t^*? 2) How do cost parameters affect IP_t^*? And, 3) how do the cost parameters affect the choice of delivery modes?

When the unit cost for the fast order increases, then it may become economic to reduce the quantity for the fast order and increase the quantity for the slow order. When the unit cost for the slow order increases, then it may become economic to reduce the slow-order quantity and increase the fast-order quantity. When the difference between the unit costs for the two orders increases, it may become economic to reduce the quantity for the fast order and increase the quantity for the slow order. We form it in the following proposition.

Proposition 1.

1) z_t^* increases as $c_{t,n}$ or p_{t,λ^-} increases;

2) z_t^* decreases as $c_{t,e} - c_{t,n}$ or $c_{t,e}$ increases; 3) z_t^* decreases as h_{t,λ^-} increases.

The following are some managerial insights on how the cost parameters affect IP_t^*. When the unit cost for the slow order increases, it may save to purchase less through the slow order. When the unit shortage cost for backordering demand which will be satisfied from the slow order increases, it may save to have fewer units that can be used for backordering. When the unit shortage cost for backordering demand in the entire period increases, it may save to have more units available for the period. When the unit holding cost for a period increases, it may save to decrease the total quantity available for the period. We form them in the following proposition.

Proposition 2.

1) IP_t^* decreases as one of $c_{t,n}$, p_{t,λ^-} and h_{t,λ^+} increases;

2) IP_t^* increases as one of p_{t,λ^+} and p_t increases.

It is intuitive that it is economic to use only the fast order when the unit cost for the fast order is not at least h_{t,λ^-} higher than the slow order, and that it is economic to use only the slow order when the unit cost for the fast order is too expensive. In either case, the minimum expected cost is convex in the beginning inventory level before the inventory decision. We form it in the following theorem.

Theorem 11.

1) If there exists t° such that $c_{t^\circ,e} + h_{t^\circ,\lambda^-} \le c_{t^\circ,n}$, then $z_{t^\circ}^* = IP_{t^\circ}^*$, and $V_{t^\circ}(\underline{z})$ is convex in $\underline{z}$.

2). If there exists t° such that $c_{t^\circ,e} \ge p_{t^\circ,\lambda^-} + c_{t^\circ,n}$, then $z_{t^\circ}^* = \underline{z}$, $IP_{t^\circ}^*$ is uniquely determined by $\dfrac{\partial V_{t^\circ}(\underline{z},\underline{z},IP)}{\partial IP} = 0$ and $V_{t^\circ}(\underline{z})$ is convex in $\underline{z}$. Furthermore, $V_{t^\circ+1}(\underline{z},z,IP)$ is unimodal in IP if $c_{t^\circ+1,n} + p_{t^\circ+1,\lambda^-} - p_{t^\circ+1} < \left(c_{t^\circ,n} + p_{t^\circ,\lambda^-}\right)$.

Theorem 11 indicates that under a mild cost condition the uniqueness of the optimal inventory level and/or inventory position in earlier periods does not depend on the number of delivery methods in future periods. In fact, part 1 does not require any further cost assumption while keeping the convexity of $V_{t^\circ}(\underline{z})$. In contrast, although part 2 keeps the convexity of $V_{t^\circ}(\underline{z})$, it changes the marginal cost in period t° to $c_{t^\circ,n} + p_{t^\circ,\lambda^-}$ from $c_{t^\circ,e}$ when the beginning inventory is less than zero. To insure the convexity of $V_{t^\circ+1}(\underline{z})$ the original cost assumption $c_{t^\circ+1,n} + p_{t^\circ+1,\lambda^-} - p_{t^\circ+1} < c_{t^\circ,e}$ for period $t^\circ+1$ is adjusted accordingly. Therefore, result 3 suggests that our model allows for the situations where for some periods there is only one delivery mode available. Fisher et al. (2001) discussed essentially a two-period variant of our problem where the first period has only the fast-delivery mode and the second period has only the slow-delivery mode.

4.5. Conclusions

Although the consideration of production (or purchase) cost drives companies to outsource their products overseas, the difficulty caused by long delivery leadtimes (due to queuing time at transfer stations, security inspections, limited transportation capacity, out-of-date distribution system, etc.) necessitates the co-use of fast delivery methods (like paying a premium to cooperate with local manufacturers, etc.). We build a stochastic, periodic-review model to examine the question of how to make optimal order decisions for the two orders at the beginning of each period. A stochastic dynamic program is formed to establish the uniqueness of the optimal quantities. Some managerial insights are derived. Recall that our model does not explicitly incorporate any specific factors causing long delivery time. Research incorporating them may be interesting.

CHAPTER 5. CONCLUSIONS

5.1. Summary

In this thesis, we have examined several inventory problems with multiple order opportunities. In particular, we have examined the newsvendor problem with two order opportunities, the stochastic multi-period inventory problem with limited order opportunities, and the stochastic multi-period problem with two supply modes. Our focus has been on studying the form of the optimal policies.

The newsvendor problem with two order opportunities in Chapter 2 examined three models differing in the timing of the second order, with the first order placed for delivery at the start of the selling season. In Model I, the second order is determined at the beginning of the season for the delivery at given time. In Model II, the second order is determined dynamically at a pre-specified time. In Model III, both the timing and quantity for the second order are determined dynamically.

For Model III, we have demonstrated, through a counterexample, that the form of the optimal policy for the second order is not necessarily of (s,S) type, therefore the first order is hard to determine. However, we were able to reveal some conditions under which the (s,S) policy is optimal and therefore the optimal first order quantity is unique, as seen in Theorem 4 and 5. For Models I and II, although having observed that the second order quantity is hard to determine in general, we have developed mild regularity conditions under which both the first and the second order quantities are easy to determine, as seen in Theorems 1, 2 and 3.

The stochastic multi-period inventory problem with limited order opportunities in Chapter 3 examined how to optimally utilize N order opportunities over T periods where $N \leq T$. We have shown that at the beginning of a period we consume an order opportunity only when the inventory level is low enough or less than a number depending on the number of periods remaining and the number of order opportunities that are still available for the remaining periods. Once an order is placed, the inventory level is raised to a number which, again, depends on the number of periods remaining and the number of order opportunities that are still available for the remaining periods. Both the order-trigger point and the order-up-to level are unique.

The stochastic multi-period inventory problem with two supply modes in Chapter 4 examined the optimal values of two order quantities placed with two suppliers, one fast and one slow, at the beginning of every period. We have shown that under reasonable conditions the form of the optimal order policy is characterized by two numbers: when the beginning inventory level is less than the smaller number, a fast order is placed to raise the inventory level to this number and a slow order is placed to raise the inventory position to the bigger number; when the beginning inventory level is between the two numbers, only a slow order is placed to raise the inventory position to the bigger number; when the beginning inventory level is higher than the bigger number, no order is placed.

5.2. Future Study

There are several limitations in this thesis. The models we discussed in Chapter 2 assumed that shortages not satisfied from on-hand inventory are accepted as backorders as far as they can be met from the delivery of the second order. The model we discussed in Chapter 3 assumed that shortages are always accepted as backorders. For these models, it would be interesting for us to explore the form of the optimal policy when shortages are lost-sales.

The fully dynamic model in Chapter 2 and the model in Chapter 3 assumed that per unit ordering cost is non-decreasing as the time gets closer to last period. It is worthwhile to relax this assumption and explore the form of the optimal policy accordingly.

The focus for the models we examined in this thesis has been on the structure of the optimal ordering policy. Future research on these models may be focused on computational studies. It is worthwhile to compare the relative value of order flexibility represented in the three models in Chapter 2. It is worthwhile to develop algorithms for efficient computation of the optimal policy.

Another limitation of our models is that we account for inventory and backorder costs at the end of the period, while the costs may actually be incurred continuously in time. Our models may need modifications similar to those in Rudi et al. (2005).

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APPENDICES

Appendix A. Proofs for Chapter 2

A.1. Proof of Lemma 1.

1) Note that $B_t^x(x_t)$ is the expected sale in the newsvendor model with a beginning inventory level x_t. Thus $B_t^x(x_t)$ is concave in x_t. Similarly, $B_t^y(y_t)$ is concave in y_t.

2) Note that $G_{L+1,T}(y_{L+1})$ is the expected newsvendor cost with a beginning inventory level y_{L+1}, composed of the expected cost for underage and overage at the end of period T. It follows that $G_{L+1,T}(y_{L+1})$ is convex in y_{L+1}.

3) Expanding the recursive equation (2), we get

$$g_1(x_1,y_1)=B_1(x_1,y_1)+\sum_{t=2}^{L}E\left[B_t\left(x_1-\xi_{1,t-1},y_1-\xi_{1,t-1}\right)\right] + \int_0^{\infty}G_{L+1,T}\left(y_1-\xi_{1,L}\right)f_{1,L}\left(\xi_{1,L}\right)d\xi_{1,L} \quad \ldots (19)$$

where $E\left[B_t\left(x_1-\xi_{1,t-1},y_1-\xi_{1,t-1}\right)\right]$ is the expectation of $B_t\left(x_1-\xi_{1,t-1},y_1-\xi_{1,t-1}\right)$ with respect to $\xi_{1,t-1}$. Then it follows, from the fact that $B_t(x_t,y_t)$ is separable in x_t and y_t, that $g_1(x_1,y_1)$ is separable in x_1 and y_1. Similarly, it follows, from the fact that $B_t(x_t,y_t)$ is convex in x_t, that $g_1(x_1,y_1)$ is convex in x_1. ■

A.2. Proof of Lemma 2.

1) It can be seen that $H_1(x_1,IP_1)$ in (P1.1) is convex in x_1 for any IP_1 since $g_1(x_1,y_1)$ is convex in x_1. It can also be seen that $H_1(x_1,x_1)$ is convex in x_1 by (19) and the fact that $B_t(x_t,x_t)=0$. This results in that $j_1(x_1)$ is piecewise convex in x_1 and that $k_1(x_1)$ is convex in x_1 from their definitions.

2) At any breakpoint IP_1^i, since $\left.\frac{\partial H_1(x_1,IP_1)}{\partial IP_1}\right|_{IP_1=IP_1^i}=0$, we have

$$\frac{dj_1(x_1)}{dx_1} = \frac{\partial H_1(x_1, IP_1)}{\partial x_1} = \frac{dk_1(x_1)}{dx_1}$$

in viewing of $\left.\frac{dk_1(x_1)}{dx_1}\right|_{x_1=IP_1^i} = \left.\frac{\partial H_1(x_1, IP_1)}{\partial x_1}\right|_{x_1=IP_1^i} + \left.\frac{\partial H_1(x_1, IP_1)}{\partial IP_1}\right|_{IP_1=IP_1^i}$.

3) By the separability in x_1 and IP_1 of $H_1(x_1, IP_1)$ from part 3) of Lemma 1 and by (1), $\frac{dj_1(x_1)}{dx_1}$ is free of IP_1 and continuously increasing in x_1 over $[0, IP_1'']$. It is also true that $\frac{dj_1(x_1)}{dx_1}$ is continuously increasing in x_1 over $[IP_1'', \infty)$ by the convexity of $H_1(x_1, x_1)$. Thus $\frac{dj_1(x_1)}{dx_1}$ is continuously increasing in x_1 over $[0, \infty)$ since $\frac{dj_1(x_1)}{dx_1}$ behaves well at IP_1'' due to $\left.\frac{\partial H_1(x_1, IP_1)}{\partial IP_1}\right|_{IP_1=IP_1''} = 0$. ■

A.3. Proof of Theorem 1.

By (P1.2) and (6), we have

$$Q_0^* = \arg\min_{Q_0 \geq 0}\left\{\min\left\{c_0Q_0 + j_1(Q_0), c_0Q_0 + k_1(Q_0)\right\}\right\}$$

which is equivalent to

$$Q_0^* = \arg\min\left\{\min_{Q_0 \geq 0}\left\{c_0Q_0 + j_1(Q_0)\right\}, \min_{Q_0 \geq 0}\left\{c_0Q_0 + k_1(Q_0)\right\}\right\}$$

Without loss of generality, we can assume that $Q_0^{***} > 0$. Thus, Q_0^{***} minimizes $c_0Q_0 + k_1(Q_0)$, which is convex by Lemma 2 part 1). We also see, from parts 2) and 3) of Lemma 2, that

$$\left.\frac{dj_1(x_1)}{dx_1}\right|_{IP_1^i} = \left.\frac{dk_1(x_1)}{dx_1}\right|_{IP_1^i} < \left.\frac{dj_1(x_1)}{dx_1}\right|_{IP_1^{i+1}} = \left.\frac{dk_1(x_1)}{dx_1}\right|_{IP_1^{i+1}}$$

and that $j_1(x_1)$ and $k_1(x_1)$ are equal at each breakpoint IP_1^i from their definitions. This implies that Q_0^{**} and Q_0^{***} fall in the same interval $[IP_1^{l-1}, IP_1^l]$ for some l:

$c_0 x_1 + j_1(x_1)$ and $c_0 x_1 + k_1(x_1)$ are minimized in the same interval; and, the solution at any breakpoint IP_1^i other than Q_0^{**} and Q_0^{***} can not be optimal. As a result, only $c_0 Q_0^{***} + k_1(Q_0^{***})$ and $c_0 Q_0^{**} + j_1(Q_0^{**})$ have to be compared to determine the initial order quantity. This leads to our conclusion. ■

A.4. Proof of Theorem 2.

1) It is sufficient for us to consider only the case $L > 0, b_1 = \cdots = b_L = 0$, for which $H_1(x_1, IP_1)$ can be written as

$$H_1(x_1, IP_1) = c_1(IP_1 - x_1) + \int_0^\infty G_{L+1,T}(IP_1 - \xi_{1,L}) f_{1,L}(\xi_{1,L}) d\xi_{1,L}$$

The convexity of $H_1(x_1, IP_1)$ with respect to IP_1 follows directly by part 2) of Lemma 1.

2) Notice the condition $L = T, b_1 = \cdots = b_{T-1} = 0, b_T > 0$ implies that the backorder cost is effectively charged independent of when the backorders occur and that $b_T \le c_u$ since otherwise the decision maker will not accept backorders. Thus, $H_1(x_1, IP_1)$ can be written as below

$$\begin{aligned} H_1(x_1, IP_1) &= c_1(IP_1 - x_1) + b_T \int_{x_1}^{IP_1} (\xi_{1,T} - x_1) f_{1,T}(\xi_{1,T}) d\xi_{1,T} + b_T \int_{IP_1}^{\infty} (IP_1 - x_1) f_{1,T}(\xi_{1,T}) d\xi_{1,T} \\ &+ c_u \int_{IP_1}^{\infty} (\xi_{1,T} - IP_1) f_{1,T}(\xi_{1,T}) d\xi_{1,T} + c_d \int_0^{IP_1} (IP_1 - \xi_{1,T}) f_{1,T}(\xi_{1,T}) d\xi_{1,T} \end{aligned}$$

With a little algebra, it is easy to show that $\dfrac{\partial^2 H_1(x_1, IP_1)}{\partial IP_1^2} = (c_u + c_d - b_T) f_{1,T}(IP_1)$.

Since $c_u \ge b_T$, it follows that $\dfrac{\partial^2 H_1(x_1, IP_1)}{\partial IP_1^2} > 0$. Therefore $H_1(x_1, IP_1)$ is convex in IP_1.

3) The backorder cost is effectively charged independent of when the backorders occur and that $b_L \le c_u$. With a little algebra, $H_1(x_1, IP_1)$ can be expressed as

$$H_1(x_1,IP_1)=c_1(IP_1-x_1)+b_L\int_{x_1}^{IP_1}(\xi_{1,L}-x_1)f_{1,L}(\xi_{1,L})d\xi_{1,L}+b_L(IP_1-x_1)\int_{IP_1}^{\infty}f_{1,L}(\xi_{1,L})d\xi_{1,L}$$
$$+c_u\int_{IP_1}^{\infty}(\xi_{1,T}-IP_1)f_{1,T}(\xi_{1,T})d\xi_{1,T}+c_d\int_0^{IP_1}(\xi_{1,T}-IP_1)f_{1,T}(\xi_{1,T})d\xi_{1,T}$$

With some more analysis, we have $\frac{\partial^2 H_1(x_1,IP_1)}{\partial IP_1^2}=-b_L f_{1,L}(IP_1)+(c_u+c_d)f_{1,T}(IP_1)$.

Recall that the monotone likelihood ratio property (MLRP) holds for $\frac{f_{1,T}(IP_1)}{f_{1,L}(IP_1)}$ over $[0,\infty)$. We see that $\frac{f_{1,T}(IP_1)}{f_{1,L}(IP_1)}$ is increasing in IP_1. If $\frac{f_{1,T}(0)}{f_{1,L}(0)}\le\frac{b_L}{c_u+c_d}$ and $\frac{f_{1,T}(\infty)}{f_{1,L}(\infty)}\ge\frac{b_L}{c_u+c_d}$, then there exists an $\widetilde{IP}_1$ such that $\frac{f_{1,T}(\widetilde{IP}_1)}{f_{1,L}(\widetilde{IP}_1)}=\frac{b_L}{c_u+c_d}$. In case $\frac{f_{1,T}(0)}{f_{1,L}(0)}>\frac{b_L}{c_u+c_d}$, $\widetilde{IP}_1$ is defined as 0; in case $\frac{f_{1,T}(\infty)}{f_{1,L}(\infty)}<\frac{b_L}{c_u+c_d}$, $\widetilde{IP}_1$ is defined as ∞. Therefore we have $\frac{\partial^2 H_1(x_1,IP_1)}{\partial IP_1^2}\le 0$ for $0\le IP_1\le\widetilde{IP}_1$ and $\frac{\partial^2 H_1(x_1,IP_1)}{\partial IP_1^2}\ge 0$ for $IP_1\ge\widetilde{IP}_1$; that is, $H_1(x_1,IP_1)$ is concave-convex in IP_1: concave on $[0,\widetilde{IP}_1]$ and convex on $[\widetilde{IP}_1,\infty)$. This leads to our conclusion.

4) To prove that $H_1(x_1,IP_1)$ is unimodal in IP_1, it is sufficient to show that $c_1+\frac{\partial g_t(x_t,y_t)}{\partial y_t}$ changes sign at most once over $(-\infty,\infty)$ for $1\le t\le L$. We do it recursively starting from $t=L$. With a little algebra, we can see that

$$c_1+\frac{\partial g_L(x_L,y_L)}{\partial y_L}=c_1+b_L\int_{y_L}^{\infty}f_L(\xi_L)d\xi_L+\int_0^{\infty}\frac{dG_{L+1,T}(y_L-\xi_L)}{dy_L}d\xi_L$$
$$=c_1+(b_L-c_u)\int_{y_L}^{\infty}f_L(\xi_L)d\xi_L+\int_0^{y_L}\frac{dG_{L+1,T}(y_L-\xi_L)}{dy_L}d\xi_L$$

By introducing a variable $u=y_L-\xi_L$, we have

$$c_1+\frac{\partial g_L(x_L,y_L)}{\partial y_L}=\int_{-\infty}^{\infty}M(u,L)f_L(y_L-u)du$$

where $M(u,L)=\begin{cases} c_1+b_L-c_u & \text{If } u<0 \\ c_1+\dfrac{\partial G_{L+1,T}(u)}{\partial u} & \text{If } u\geq 0 \end{cases}$. It is easy to see that $M(u,L)$ changes sign at most once over $(-\infty,\infty)$ in view that $G_{L+1,T}$ is convex, $\left.\dfrac{\partial G_{L+1,T}(u)}{\partial u}\right|_{u=0}=-c_u$, $\lim_{u\to\infty}\dfrac{\partial G_{L+1,T}(u)}{\partial u}=c_d$ and $c_1+b_L-c_u\leq 0$. As a result, $c_1+\dfrac{\partial g_L(x_L,y_L)}{\partial y_L}$ changes sign at most once over $(-\infty,\infty)$ with respect to y_L, and $\left.\dfrac{\partial g_L(x_L,y_L)}{\partial y_L}\right|_{y_L=0}=c_1+b_L-c_u\leq 0$.

Suppose $c_1+\dfrac{\partial g_{t+1}(x_{t+1},y_{t+1})}{\partial y_{t+1}}$ changes sign at most once and has a non-positive value of $c_1+\sum_{j=t+1}^{L}b_j-c_u$ at $y_{t+1}=0$. For $c_1+\dfrac{\partial g_t(x_t,y_t)}{\partial y_t}$, we have

$$c_1+\frac{\partial g_t(x_t,y_t)}{\partial y_t}=c_1+\left(\sum_{j=t}^{L}b_j-c_u\right)\int_{y_t}^{\infty}f_t(\xi_t)d\xi_t+\int_0^{y_t}\frac{\partial g_{t+1}(x_t-\xi_t,y_t-\xi_t)}{\partial y_t}f_t(\xi_t)d\xi_t$$

$$=\int_{-\infty}^{\infty}M(u,t)f_t(y_t-u)du \qquad \text{where}$$

$$M(u,t)=\begin{cases} c_1+\sum_{j=t}^{L}b_j-c_u & \text{If } u<0 \\ c_1+\left.\dfrac{\partial g_{t+1}(x_t+u-y_t,v)}{\partial v}\right|_{v=u} & \text{If } u\geq 0 \end{cases}$$

It can be seen that $M(u,t)$ changes sign at most once over $(-\infty,\infty)$ since $M(u,t)\leq 0$ for $u<0$ and $c_1+\left.\dfrac{\partial g_{t+1}(x_t+u-y_t,v)}{\partial v}\right|_{v=u}$ changes sign at most once over $[0,\infty)$ from "-" to "+". Therefore $c_1+\dfrac{\partial g_t(x_t,y_t)}{\partial y_t}$ changes sign at most once and $c_1+\left.\frac{\partial g_t(x_t,y_t)}{\partial y_t}\right|_{y_t=0}=c_1+\sum_{j=t}^{L}b_j-c_u\le 0$. By induction, the conclusion is established.

5) To prove $H_1(x_1,IP_1)$ has at most one local minimum and one local maximum with respect to IP_1, it is sufficient to show that $c_1+\frac{\partial g_t(x_t,y_t)}{\partial y_t}$ changes sign at most twice over $(-\infty,\infty)$ for $1\le t\le L$. If $c_1+\sum_{j=1}^{L}b_j-c_u\le 0$, then the result is evident by part 4) since PF_2 includes PF_3. Next we assume $c_1+\sum_{j=1}^{L}b_j-c_u>0$. Define $t^*=\max\left\{1\le t\le L \middle| c_1+\sum_{j=t}^{L}b_j-c_u>0\right\}$. By part 4), $c_1+\frac{\partial g_t(x_t,y_t)}{\partial y_t}$ changes sign at most once over $(-\infty,\infty)$ for $t>t^*$. We next show that, for any $t\le t^*$, $c_1+\frac{\partial g_t(x_t,y_t)}{\partial y_t}$ changes sign at most twice over $(-\infty,\infty)$. We do so recursively starting from $t=t^*$. $c_1+\frac{\partial g_{t^*}(x_{t^*},y_{t^*})}{\partial y_{t^*}}=\int_{-\infty}^{\infty}M(u,t^*)f_{t^*}(y_{t^*}-u)du$, where

$$M(u,t^*)=\begin{cases} c_1+\sum_{j=t^*}^{L}b_j-c_u & \text{If } u<0 \\ c_1+\left.\frac{\partial g_{t^*+1}(x_{t^*}+u-y_{t^*},v)}{\partial v}\right|_{v=u} & \text{If } u\ge 0 \end{cases}$$

. It can be seen that $M(u,t^*)$ changes sign at most twice over $(-\infty,\infty)$ since $M(u,t^*)>0$ for $u<0$, and $c_1+\left.\frac{\partial g_{t^*+1}(x_{t^*}+u-y_{t^*},v)}{\partial v}\right|_{v=u}$ changes sign at most once over $[0,\infty)$ starting from non-positive. Therefore $c_1+\frac{\partial g_{t^*}\left(x_{t^*},y_{t^*}\right)}{\partial y_{t^*}}$ changes sign at most twice and has a positive value of $c_1+\sum_{j=t^*}^{L}b_j-c_u$ at $y_{t^*}=0$. For t^*-1, we have:

$$c_1+\frac{\partial g_{t^*-1}\left(x_{t^*-1},y_{t^*-1}\right)}{\partial y_{t^*-1}}=\int_{-\infty}^{\infty}M\left(u,t^*-1\right)f_{t^*-1}\left(y_{t^*-1}-u\right)du \quad \text{where}$$

$$M\left(u,t^*-1\right)=\begin{cases} c_1+\sum_{j=t^*-1}^{L}b_j-c_u & \text{If } u<0 \\ c_1+\left.\frac{\partial g_{t^*}\left(x_{t^*-1}+u-y_{t^*-1},v\right)}{\partial v}\right|_{v=u} & \text{If } u\geq 0 \end{cases}$$

It can be seen that $M\left(u,t^*-1\right)$ changes sign at most twice over $\left(-\infty,\infty\right)$ since $M\left(u,t^*-1\right)>0$ for $u<0$, and $c_1+\left.\frac{\partial g_{t^*}\left(x_{t^*-1}+u-y_{t^*-1},v\right)}{\partial v}\right|_{v=u}$ changes sign at most twice over $\left[0,\infty\right)$ starting from "+". Therefore $c_1+\frac{\partial g_{t^*-1}\left(x_{t^*-1},y_{t^*-1}\right)}{\partial y_{t^*-1}}$ changes sign at most twice and has a positive value of $c_1+\sum_{j=t^*-1}^{L}b_j-c_u$ at $y_{t^*-1}=0$. By induction, part 5) follows.■

A.5. Proof of Theorem 3.

1) Analogous to part 1) of Theorem 2, it is true that $H_\tau\left(x_\tau,IP_\tau\right)$ is convex in IP_τ, where

$$H_\tau\left(x_\tau,IP_\tau\right)=c_\tau\left(IP_\tau-x_\tau\right)+\int_0^{\infty}G_{\tau+L,T}\left(IP_\tau-\xi_{\tau,\tau+L-1}\right)f_{\tau,\tau+L-1}\left(\xi_{\tau,\tau+L-1}\right)d\xi_{\tau,\tau+L-1}$$

Thus, a base-stock policy $Q_\tau=\max\left\{IP_\tau^*-x_\tau,0\right\}$ is the optimal ordering rule at the beginning of period τ, where $IP_\tau^*\geq 0$ is the value of IP_τ that attains the minimum of $H_\tau\left(x_\tau,IP_\tau\right)$ over $\left[0,\infty\right)$, leading to an expression for $h_\tau\left(x_\tau\right)$ below:

$$h_\tau(x_\tau) = \begin{cases} H_\tau(x_\tau, IP_\tau^*) & \text{if } x_\tau < IP_\tau^* \\ H_\tau(x_\tau, x_\tau) & \text{if } x_\tau \geq IP_\tau^* \end{cases}$$

The convexity of $h_\tau(x_\tau)$ is implied by the convexity of $H_\tau(x_\tau, IP_\tau^*)$ and $H_\tau(x_\tau, x_\tau)$ with respect to x_τ (analogous to $H_1(x_1, IP_1)$ and $H_1(x_1, x_1)$, respectively), and by the inequality $\frac{\partial H_\tau(x_\tau, IP_\tau)}{\partial IP_\tau} \geq 0$ for $IP_\tau > IP_\tau^*$.

To show that $TC_{II}(Q_0)$ is convex in Q_0, it is sufficient to show that $h_1(x_1)$ is convex in x_1. In view of the recursive equation (7), it is then sufficient to show that $h_2(x_2)$ is convex in x_2 since $b_1 \int_{x_1}^{\infty}(\xi_1 - x_1) f_1(\xi_1) d\xi_1$ is convex in x_1, where $x_2 = x_1 - \xi_1$. The rationale for this is that a convolution transformation of a convex function is a convex function and a summation of two convex functions is a convex function. Following the same logic, it is sufficient to show that $h_\tau(x_\tau)$ is convex in x_τ, which we have proved above.

2) Analogous to part 2) of Theorem 2 it is true that $H_\tau(x_\tau, IP_\tau)$ is convex in IP_τ, where

$$H_\tau(x_\tau, IP_\tau) = c_\tau(IP_\tau - x_\tau) + b_T \int_{x_\tau}^{IP_\tau}(\xi_{\tau,T} - x_\tau) f_{\tau,T}(\xi_{\tau,T}) d\xi_{\tau,T} + b_T \int_{IP_\tau}^{\infty}(IP_\tau - x_\tau) f_{\tau,T}(\xi_{\tau,T}) d\xi_{\tau,T}$$
$$+ c_u \int_{IP_\tau}^{\infty}(\xi_{\tau,T} - IP_\tau) f_{\tau,T}(\xi_{\tau,T}) d\xi_{\tau,T} + c_d \int_0^{IP_\tau}(IP_\tau - \xi_{\tau,T}) f_{\tau,T}(\xi_{\tau,T}) d\xi_{\tau,T}$$

The proof for the convexity of $TC_{II}(Q_0)$ in Q_0 is similar to the proof of part 1) of Theorem 3 above.

3) With a little algebra, $H_\tau(x_\tau, IP_\tau)$ can be expressed as

$$H_\tau(x_\tau, IP_\tau) = c_\tau(IP_\tau - x_\tau) + b_{\tau+L-1} \int_{x_\tau}^{IP_\tau}(\xi_{\tau,\tau+L-1} - x_\tau) f_{\tau,\tau+L-1}(\xi_{\tau,\tau+L-1}) d\xi_{\tau,\tau+L-1}$$
$$+ b_{\tau+L-1}(IP_\tau - x_\tau) \int_{IP_\tau}^{\infty} f_{\tau,\tau+L-1}(\xi_{\tau,\tau+L-1}) d\xi_{\tau,\tau+L-1}$$
$$+ c_u \int_{IP_\tau}^{\infty}(\xi_{\tau,T} - IP_\tau) f_{\tau,T}(\xi_{\tau,T}) d\xi_{\tau,T} + c_d \int_0^{IP_\tau}(\xi_{\tau,T} - IP_\tau) f_{\tau,T}(\xi_{\tau,T}) d\xi_{\tau,T}$$

Analogous to part 3) of Theorem 2 it can be seen that $H_\tau(x_\tau, IP_\tau)$ is concave-convex in IP_τ.

4) To prove that $H_\tau(x_\tau, IP_\tau)$ is unimodal in IP, it is sufficient to show that $c_\tau + \frac{\partial g_t(x_t, y_t)}{\partial y_t}$ change sign at most once over $(-\infty, \infty)$ for $\tau \le t \le \tau + L - 1$. It can be done similar to the proof of Theorem 2 part 4). The proof for convexity of $TC_{II}(Q_0)$ in Q_0 is similar to the proof of part 1) of Theorem 3 above.

5) To prove that $H_\tau(x_\tau, IP_\tau)$ has at most one local minimum and one local maximum with respect to IP, it is sufficient to show that $c_\tau + \frac{\partial g_t(x_t, y_t)}{\partial y_t}$ changes sign at most twice over $(-\infty, \infty)$ for $\tau \le t \le \tau + L - 1$. It can be done similar to how it was done in the proof of Theorem 2 part 5). ■

A.6. Proof of Theorem 4.

1) The outline for the proof is: we first show that for big-enough beginning inventory level x_t, $j_t(x_t) \ge k_t(x_t)$; then we show that $j_t(x_t)$ and $k_t(x_t)$ cross only once so that an (s, S) policy is the optimal rule for whether to place the second order and how much when ordering at the beginning of period t. By the definition of $H_t(x_t, IP_t)$ and $L = 0$, it can be seen that

$$H_t(x_t, IP_t) = c_t(IP_t - x_t) + c_u \int_{IP_t}^{\infty} (\xi_{t,T} - IP_t) f_{t,T}(\xi_{t,T}) d\xi_t + c_d \int_0^{IP_t} (IP_t - \xi_{t,T}) f_{t,T}(\xi_{t,T}) d\xi_t$$

It can be verified that for any t, $H_t(x_t, IP_t)$ is convex in IP_t with limits $\lim_{IP_t \to \infty} H_t(x_t, IP_t) = \infty$ and $\lim_{IP_t \to -\infty} H_t(x_t, IP_t) = \infty$. IP_t^* is the unique minimizer of $H_t(x_t, IP_t)$ as a function of IP_t. When $x_t \ge IP_t^*$, if an order is forced to be placed then the optimal order size is zero by the convexity of $H_t(x_t, IP_t)$ with respect to IP_t. Therefore $j_t(x_t) \ge k_t(x_t)$ holds true when $x_t \ge IP_t^*$ since the $k_t(x_t)$ solution still keeps an order opportunity, while starting with the same inventory level as the $j_t(x_t)$ solution.

We next focus on the case when $x_t < IP_t^*$. From (11) and the expression for $H_t(x_t, IP_t)$ above, we get

$$j_t(x_t) = -c_t x_t + c_t IP_t^* + c_u \int_{IP_t^*}^{\infty} \left(\xi_{t,T} - IP_t^*\right) f_{t,T}\left(\xi_{t,T}\right) d\xi_{t,T} + c_d \int_0^{IP_t^*} \left(IP_t^* - \xi_{t,T}\right) f_{t,T}\left(\xi_{t,T}\right) d\xi_{t,T}$$

Thus, to compare $j_t(x_t)$ and $k_t(x_t)$, it is sufficient to compare Θ_t and $K_t(x_t) = k_t(x_t) + c_t x_t$, where

$$\Theta_t = c_t IP_t^* + c_u \int_{IP_t^*}^{\infty} \left(\xi_{t,T} - IP_t^*\right) f_{t,T}\left(\xi_{t,T}\right) d\xi_{t,T} + c_d \int_0^{IP_t^*} \left(IP_t^* - \xi_{t,T}\right) f_{t,T}\left(\xi_{t,T}\right) d\xi_{t,T}.$$

(Note for $x_t \geq IP_t^*$, $K_t(x_t) \leq \Theta_t$.) The question that needs to be answered next is: Given that we are at the beginning of a period t and the inventory level $x_t < IP_t^*$, for what values of x_t is it optimal to place an order? We answer this question in a recursive manner by starting from $t = T$.

Case $t = T$

It is easy to see that $k_T(x_T) = H_T(x_T, x_T)$, thus $K_T(x_T)$ is convex and minimized at IP_T^*, and $\Theta_T \leq K_T(x_T)$ for $x_T \leq IP_T^*$. Therefore the optimal policy at the beginning of period T is: If $x_T \leq IP_T^*$ then it is optimal to raise the inventory level to IP_T^*. As a result, $h_T(x_T)$ is convex in x_T.

Before further discussion we state a lemma which will be used later.

Lemma 7. $K_t'(x_t) < 0$ for $x_t \leq 0$ and $t \leq T-1$.

Proof "$x_t \leq 0$" implies that there must be an order placement in one of the remaining $T-t$ periods by the backordering assumption and the assumption

$c_u > \max\{c_1, c_2, \cdots, c_T\}$. Suppose it is placed in period t_1 ($t_1 > t$) with a unit price of c_{t_1}. Then we have

$$\begin{aligned} K_t'(x_t) &= c_t - b_t \int_{x_t}^{\infty} f_t(\xi_t)\, d\xi_t + \int_0^{\infty} h_{t+1}'(x_t - \xi_t) f_t(\xi_t)\, d\xi_t \\ &= c_t - b_t - \cdots - b_{t_1 - 1} - c_{t_1} < 0 \end{aligned} \quad \blacksquare$$

Knowing that $K_t(IP_t^*) \le \Theta_t$, Lemma 7 suggests that Θ_t and $K_t(x_t)$ cross at least once over $(-\infty, IP_t^*]$ (note here Θ_t is a constant function with respect to x_t). In what follows we show that they cross once.

Case $t = T-1$

To show that Θ_{T-1} and $K_{T-1}(x_{T-1})$ cross once over $(-\infty, IP_{T-1}{}^*]$, it is sufficient to show that $K_{T-1}(x_{T-1})$ is convex in x_{T-1}. Notice that

$$K_{T-1}(x_{T-1}) = c_{T-1} y + b_{T-1} \int_{x_{T-1}}^{\infty} (\xi_{T-1} - x_{T-1}) f_{T-1}(\xi_{T-1})\, d\xi_{T-1} + \int_0^{\infty} h_T(x_{T-1} - \xi_{T-1}) f_{T-1}(\xi_{T-1})\, d\xi_{T-1}$$

The convexity of $K_{T-1}(x_{T-1})$ follows because each term in the R.H.S. is convex in x_{T-1}.

To show that Θ_t and $K_t(x_t)$ cross once for all $t \le T-2$, we show that $K_t(x_t)$ is unimodal for $t \le T-2$. Notice that

$$\begin{aligned} K_t'(K_t'(x_t) &= c_t - b_t \int_{x_t}^{\infty} f_t(\xi_t)\, d\xi_t + \int_0^{\infty} h_{t+1}'(x_t - \xi_t) f_t(\xi_t)\, d\xi_t \\ &= \int_{-\infty}^{\infty} M_t(u) f_t(x_t - u)\, du \end{aligned} \quad \text{.... (13)}$$

where

$$M_t(u) = \begin{cases} c_t - b_t + h_{t+1}'(u) & \text{if } u \le 0 \\ c_t + h_{t+1}'(u) & \text{if } u > 0 \end{cases}$$

To prove that $K_t(x_t)$ is unimodal, it is sufficient to show that $K_t'(x_t)$ changes sign once over $(-\infty,\infty)$. It is then sufficient to show that $M_t(u)$ changes sign once over $(-\infty,\infty)$ since f_t is a PF_2 density. We first discuss the special case $t=T-2$; Then we discuss the general case by induction.

Case $t=T-2$

We follow two steps to prove that $M_{T-2}(y)$ changes sign once over $(-\infty,\infty)$.

Step 1: Claim $M_{T-2}(y)$ is negative when $y\le 0$. In fact, if $y\le 0$, we have

$$M_{T-2}(y)=\begin{cases} c_{T-2}-b_{T-2}+j_{T-1}'(y) & \text{if } s_{T-1}>0 \\ c_{T-2}-b_{T-2}+j_{T-1}'(y) & \text{if } s_{T-1}\le 0 \;\&\; y<s_{T-1} \\ c_{T-2}-b_{T-2}+k_{T-1}'(y) & \text{if } s_{T-1}\le 0 \;\&\; y\ge s_{T-1} \end{cases}$$
$$=\begin{cases} c_{T-2}-p_{T-2}-c_{T-1} & \text{if } s_{T-1}>0 \\ c_{T-2}-p_{T-2}-c_{T-1} & \text{if } s_{T-1}\le 0 \;\&\; y<s_{T-1} \\ c_{T-2}-p_{T-2}-c_{T-1}+K_{T-1}'(y) & \text{if } s_{T-1}\le 0 \;\&\; y\ge s_{T-1} \end{cases}$$
$$<0$$

where s_{T-1} is the order-triggering point in period $T-1$.

Step 2: $M_{T-2}(y)$ changes sign at most once over $(0,\infty)$ from "-" to "+".

Without loss of generality, we can only discuss the case $s_{T-1}>0$ since the opposite case can be discussed similarly. Notice that

$$c_{T-2}+h_{T-1}'(y)=\begin{cases} c_{T-2}-c_{T-1} & \text{if } y<s_{T-1} \\ c_{T-2}-c_{T-1}+K_{T-1}'(y) & \text{if } y\ge s_{T-1} \end{cases}$$

$c_{T-2}+h_{T-1}'(y)$ is negative if $y<s_{T-1}$; For $y\ge s_{T-1}$, $c_{T-2}+h_{T-1}'(y)$ increases to a positive value of $c_{T-2}+c_d$ as y approaches ∞ by the convexity of $K_{T-1}(y)$, which follows from the convexity of $h_T(x_T)$. Therefore $c_{T-2}+h_{T-1}'(y)$ changes sign at most once as y traverses from 0 to ∞.

Similarly, we can show that $c_{T-3}-c_{T-2}+M_{T-2}(y)$ changes sign once over $(-\infty,\infty)$ from "-" to "+".

Case General t

Analogous to the case $t=T-2$, we follow two steps to show that $M_t(y)$ changes signs once.

Step 1: Claim $M_t(y)$ is negative when $y\le 0$. A logic similar to that for case $\tau=T-2$ can be applied.

Step 2: $M_t(y)$ changes sign at most once over $(0,\infty)$ from "-" to "+".

Without loss of generality, we can only discuss the case $s_{t+1}>0$, since the opposite case can be discussed similarly. We will discuss the sign of $M_t(y)$ for different categories of y.

For $0<y<s_{t+1}$, $M_t(y)=c_t+h_{t+1}'(y)=c_t+j_{t+1}'(y)=c_t-c_{t+1}\le 0$. For $s_{t+1}\le y$

$$\begin{aligned} M_t(y) &= c_t+h_{t+1}'(y)=c_t+k_{t+1}'(y) \\ &= c_t-c_{t+1}+K_{t+1}'(y) \\ &= \int_{-\infty}^{\infty}\left(c_t-c_{t+1}+M_{t+1}(u)\right)f_{t+1}(y-u)\,du \end{aligned}$$

Since $c_t-c_{t+1}+M_{t+1}(u)$ changes sign once over $(-\infty,\infty)$, $M_t(y)$ changes sign at most once over $[s_{t+1},\infty)$. In particular, if it does change sign once then the sign changes from negative to positive; Otherwise the sign remains positive. For either case, $M_t(y)$ changes sign at most once over $(0,\infty)$.

To summarize, for general t, we show that $M_t(y)$ changes sign once from negative to positive as y traverses from $-\infty$ to ∞. Similarly, we can show that $c_{t-1}-c_t+M_t(u)$ changes sign once over $(-\infty,\infty)$ from "-" to "+".

By induction, we have shown that $K_t(x_t)$ is unimodal for $t\le T-2$. Therefore there exists a unique $x_t=s_t<IP_t^*$ such that $K_t(x_t)$ intersects with Θ_t, i.e., an (s,S) policy with $s=s_t, S=IP_t^*$ is optimal.

2) Since $L>0, b_1=\cdots=b_{T-1}=0, b_T\geq 0$, period $T-L+1$ is the last period when an order can be placed. Moreover, we can see that $H_t(x_t, IP_t)$ has an expression below

$$H_t(x_t, IP_t)=$$
$$\begin{cases} c_t(IP_t-x_t)+b_T\int_{x_t}^{IP_t}(\xi_{t,T}-x_t)f_{t,T}(\xi_{t,T})d\xi_t+b_T\int_{IP_t}^{\infty}IP_tf_{t,T}(\xi_{t,T})d\xi_t & \\ \quad +c_u\int_{IP_t}^{\infty}(\xi_{t,T}-IP_t)f_{t,T}(\xi_{t,T})d\xi_t+c_d\int_0^{IP_t}(IP_t-\xi_{t,T})f_{t,T}(\xi_{t,T})d\xi_t & t=T-L+1 \\ c_t(IP_t-x_t)+c_u\int_{IP_t}^{\infty}(\xi_{t,T}-IP_t)f_{t,T}(\xi_{t,T})d\xi_t+c_d\int_0^{IP_t}(IP_t-\xi_{t,T})f_{t,T}(\xi_{t,T})d\xi_t & t<T-L+1 \end{cases}$$

$k_t(x_t)$ can be recursively expressed as $k_t(x_t)=\int_0^{\infty}h_{t+1}(x_t-\xi_t)f_t(\xi_t)d\xi_t$ for $t<T-L+1$, and expressed as $k_t(x_t)=H_t(x_t,x_t)$ for $t=T-L+1$. Then the approach in the proof for part 1) of Theorem 4 can be applied to establish the conclusion. ■

A.7. Proof of Theorem 5.

It is sufficient to show that $TC_{III}(Q_0)$ is unimodal. Since $TC_{III}(Q_0)$ has a negative slope c_0-c_1 for $Q_0<s_1$, it remains to show that $TC_{III}(Q_0)$ is unimodal over (s_1,∞). This is true since $TC_{III}(Q_0)$, with an expression below

$$TC_{III}(Q_0)=c_0Q_0+b_1\int_{Q_0}^{\infty}(\xi_1-Q_0)f_1(\xi_1)d\xi_1+\int_0^{\infty}h_2(Q_0-\xi_1)f_1(\xi_1)d\xi_1$$

has a structure similar to $k_1(x_1)$, which we have shown is unimodal. ■

Appendix B. Omitted Proofs in Chapter 3

B.1. Proof of Theorem 6

Recall the definition of $Đ_1(\tau,y)$. We have

$$Đ_1'(\tau,y)=c_\tau-p\int_y^\infty f_\tau(\xi)d\xi+\int_0^\infty V_1'(\tau-1,y-\xi)f_\tau(\xi)d\xi$$
$$=\int_{-\infty}^\infty M(\tau,u)f_\tau(y-u)du$$

where

$$M(\tau,u)=\begin{cases}c(\tau)-p+V_1'(\tau-1,u) & \text{if } u\le 0\\ c(\tau)+V_1'(\tau-1,u) & \text{if } u>0\end{cases}$$

To prove that $Đ_1(\tau,y)$ is unimodal, it is sufficient to show that $Đ_1'(\tau,y)$ changes sign once over $(-\infty,\infty)$. It is then sufficient to show that $M(\tau,y)$ changes sign once over $(-\infty,\infty)$. Before our discussion of these cases, we first state a lemma which will be used later.

Lemma 8. $Đ_1'(\tau,y)<0$ for $y\le 0$ and $\tau\ge 2$.

Proof "$y\le 0$ and demand greater than zero" implies that there must be an order placement in one of the remaining $\tau-1$ periods by the backordering assumption and the assumption $p_1>c_1$. Suppose it is placed at time $T-\tau_1+1$ ($\tau_1<\tau$) with unit price of c_{τ_1}. Then we have

$$Đ_1'(\tau,y)=c_\tau-p\int_y^\infty f_\tau(\xi)d\xi+\int_0^\infty V_1'(\tau-1,y-\xi)f_\tau(\xi)d\xi$$
$$=c_\tau-p+\int_0^\infty V_1'(\tau-1,y-\xi)f_\tau(\xi)d\xi \qquad \blacksquare$$
$$=c_\tau-(\tau-\tau_1)p-c_{\tau_1}<0$$

As in the proof of Theorem 4, we can prove

1) $M(\tau,u)$ changes sign once over $(-\infty,\infty)$ from "-" to "+".

2) $c_{\tau+1}-c_\tau+M(\tau,u)$ changes sign once over $(-\infty,\infty)$ from "-" to "+".

Therefore $Đ_1'(\tau,y)$ changes sign once and $Đ_1(\tau,y)$ is unimodal. Theorefore we can claim that there exists a unique $y=y^*(\tau)<IP^*(\tau)$ such that $Đ_1(\tau,y)$ intersects with $\Theta(\tau)$, i.e., an (s,S) policy with $s=y^*(\tau), S=IP^*(\tau)$ is optimal for τ. The unimodality of $V_1(\tau,\text{y})$ is also seen from our discussion above. This completes our proof for Theorem 6. ■

B.2. Proof of Theorem 7

We show that there exists $y=y_k^*(\tau)\le IP_k^*(\tau)$ such that $Đ_k(\tau,y)=\Theta_k(\tau)$ at $y_k^*(\tau)$, and that $Đ_k(\tau,y)$ stays below $\Theta_k(\tau)$ for $y\in(y_k^*(\tau), IP_k^*(\tau))$ and stays above $\Theta_k(\tau)$ for $y\in(-\infty, y_k^*(\tau))$. The existence of $y_k^*(\tau)$ where $Đ_k(\tau,y)$ and $\Theta_k(\tau)$ intersect follows from the following lemma.

Lemma 9.

1) $Đ_k'(\tau,y)<0$ for $y\le 0$ and $\tau\ge 2$.

2) $\lim_{y\to-\infty} Đ_k(\tau,y)=\infty$; $\lim_{y\to\infty} Đ_k'(\tau,y)=c_\tau$.

3) $Đ_k(\tau, IP^*(\tau))\le\Theta_k(\tau)$.

Proof The proof can be done, similar to the proofs for Lemma 3, Lemma 1 and Lemma 2. ■

To show that $Đ_k(\tau,y)$ stays below $\Theta_k(\tau)$ for $y\in(y_k^*(\tau), IP_k^*(\tau))$, it is sufficient to show that $Đ_k(\tau,y)$ is unimodal in y. Similar to $Đ(\tau,y)$, we have

$$Đ_k'(\tau,y)=\int_{\infty}^{\infty} M_k(\tau,u) f_\tau(y-u)\,du$$

where $M_k(\tau,u)=\begin{cases} c_\tau - p + V_k'(\tau-1,u) & \text{if } u\le 0 \\ c_\tau + V_k'(\tau-1,u) & \text{if } u>0 \end{cases}$. It is then sufficient to show that $M_k(\tau,u)$ changes sign once.

In what follows, we first discuss three special cases: $\tau \le k, \tau = k+1, k+2$; then a similar discussion can be applied to the general case.

Case $\tau \le k$

In this case, the number of order opportunities is greater than or equal to the number of periods. Without loss of generality, we only discuss the case $\tau = k$. That is, the decision maker has an order opportunity at the beginning of each period. Then this becomes one of the traditional τ-period, stochastic inventory problems. It can be seen that $Đ_k(k,y)$ is convex in y. Therefore, a base-stock policy is optimal for any remaining period. That is, for $\tau = k$ there exists a unique positive real number, $IP_\tau^*(\tau)$, such that if the inventory level is less than $IP_\tau^*(\tau)$, then an order is placed to raise inventory level up to $IP_\tau^*(\tau)$; otherwise no order is placed.

Case $\tau = k+1$

We present Lemma 10 below to show that

$$M_k(k+1,y)=\begin{cases} c_{k+1} - p + V_k'(k,y) & \text{if } y\le 0 \\ c_{k+1} + V_k'(k,y) & \text{if } y>0 \end{cases}$$

changes sign once.

Lemma 10.

1) $M_k(k+1,y)$ changes sign once over $(-\infty,\infty)$ from "-" to "+".

2) $c_{k+2} - c_{k+1} + M_k(k+1,y)$ changes sign once over $(-\infty,\infty)$ from "-" to "+".

Proof 1) We need two steps to prove that $M_k(k+1,y)$ changes sign once over $(-\infty,\infty)$.

Step 1: Claim $M_k(k+1,y)$ is negative when $y \le 0$. Similar to the case involving general τ in the proof for Theorem 4, we can see that $M_k(k+1,y)$ is $c_{k+1}-p-c_k+Đ_k^{'}(k,y)$ if $y_k^*(k) \le 0$ and $y \ge y_k^*(k)$, and is $c_{k+1}-p-c_k$ otherwise. For either case $M_k(k+1,y)$ is negative.

Step 2: $M_k(k+1,y)$ changes sign at most once over $(0,\infty)$ from "-" to "+".

Without loss of generality, we only need to discuss the case $y_k^*(k) > 0$ since the opposite case can be discussed similarly. Notice that

$$c_{k+1}+V_k^{'}(k,y)=\begin{cases} c_{k+1}-c_k & \text{if } y < y_k^*(k) \\ c_{k+1}-c_k+Đ_k^{'}(k+1,y) & \text{if } y \ge y_k^*(k) \end{cases}$$

$c_{k+1}+V_k^{'}(k,y)$ is negative if $y < y_k^*(k)$; For $y \ge y_k^*(k)$, $c_{k+1}+V_k^{'}(k,y)$ increases to a positive value of c_{k+1} as y approaches ∞ by the convexity of $Đ_k(k,y)$. Therefore $c_{k+1}+V_k^{'}(k,y)$ changes sign at most once as y traverses from 0 to ∞.

Therefore, the claim that $M_k(k+1,y)$ changes sign once over $(-\infty,\infty)$ from "-" to "+" follows from the results of step 1 and step 2.

2) It follows by a similar logic. ■

By Lemma 10, we can claim that $Đ_k^{'}(k+1,y)$ changes sign once and that $Đ_k(k+1,y)$ is unimodal. There exists a unique $y=y_k^*(k+1) < IP_k^*(k+1)$ such that $Đ_k(k+1,y)$ intersects with $\Theta_k(k+1)$ i.e., an (s,S) policy with $s=y_k^*(k+1), S=IP_k^*(k+1)$ is optimal for $\tau=k+1$.

Case $\tau=k+2$

We will prove that an (s,S) policy is optimal for this case by showing that $Đ_k(k+2,y)$ is unimodal. It is sufficient to show that $Đ_k^{'}(k+2,y)$ changes sign once. We achieve it by proving $M_k(k+2,y)=\begin{cases} c_{k+2}-p+V_k^{'}(k+1,y) & \text{if } y\le 0 \\ c_{k+2}+V_k^{'}(k+1,y) & \text{if } y>0 \end{cases}$ changes sign once over $(-\infty,\infty)$ through two steps below.

Step 1: Claim $M_k(k+2,y)$ is negative when $y\le 0$. This can be done analogous to the case of general τ in the proof for Theorem 4.

Step 2: $M_k(k+2,y)$ changes sign at most once over $(0,\infty)$ from "-" to "+".

Without loss of generality, we only need to discuss the case $y_k^*(k+1)>0$ since the opposite case can be discussed similarly. We proceed by discussing the sign of $M_k(k+2,y)$ for different categories of $y(>0)$. For $0<y<y_k^*(k+1)$, $M_k(k+2,y)=c_{k+2}+O_k^{'}(k+1,y)=c_{k+2}-c_{k+1}\le 0$.

For $y\ge y_k^*(k+1)$

$$\begin{aligned} M_k(k+2,y) &= c_{k+2}+D_k^{'}(k+1,y) \\ &= c_{k+2}-c_{k+1}+Đ_k^{'}(k+1,y) \\ &= \int_{-\infty}^{\infty}\left(c_{k+2}-c_{k+1}+M_k(k+1,u)\right)f_{k+1}(y-u)\,du \end{aligned}$$

where $c_{k+2}-c_{k+1}+M_k(k+1,u)=\begin{cases} c_{k+2}-p+V_k^{'}(k,u) & \text{if } u\le 0 \\ c_{k+2}+V_k^{'}(k,u) & \text{if } u>0 \end{cases}$. Note the fact that $c_{k+2}-c_{k+1}+M_k(k+1,u)$ changes sign once over $(-\infty,\infty)$ from "-" to "+" (by induction) implies that $\int_{-\infty}^{\infty}\left(c_{k+2}-c_{k+1}+M_k(k+1,u)\right)f_{k+1}(y-u)\,du$, as a function of y, changes sign once over $(-\infty,\infty)$ from "-" to "+". As a result $M_k(k+2,y)$ changes sign at most once over $(y_k^*(k+1),\infty)$. In particular, if it does change sign once then the sign must change from "-" to "+"; otherwise the sign stays "+".

To summarize for the case $\tau = k+2$, we show that $M_k(k+2,y)$ changes sign once from negative to positive as y traverses from $-\infty$ to ∞. We form it in the following lemma.

Lemma 11.

1) $M_k(k+2,y)$ changes sign once over $(-\infty,\infty)$ from "-" to "+".

2) $c_{k+3} - c_{k+2} + M_k(k+2,y)$ changes sign once over $(-\infty,\infty)$ from "-" to "+".

Thus, $Ð_k^{'}(k+2,y)$ changes sign once and $M_k(k+2,y)$ is unimodal. Now we claim that there exists a unique $y = y_k^*(k+2) < IP_k^*(k+2)$ such that $Ð_k(k+2,y)$ intersects with $\Theta_k(k+2)$ at $y = y_k^*(k+2)$, i.e., an (s,S) policy with $s = y_k^*(k+2), S = IP_k^*(k+2)$ is optimal.

Applying similar logic to the proof for the general case in the single order opportunity case, we can show that for $\tau \geq k+1$, we have 1) $Ð_k(\tau,y)$ is unimodal and $Ð_k^{'}(\tau,y)$ changes sign once from negative to positive as y traverse from $-\infty$ to ∞; 2) There exists a unique $y_k^*(\tau) < IP_k^*(\tau)$ such that

$$V_k(\tau,y) = \begin{cases} O_k(\tau,y) & \text{for } y < y_k^*(\tau) \\ D_k(\tau,y) & \text{for } y \geq y_k^*(\tau) \end{cases}$$

3) $c_{\tau+1} - c_\tau + M_k(\tau,u)$ changes sign once over $(-\infty,\infty)$.

The unimodality of $V_k(\tau,y)$ is seen from our discussion above. This completes our proof for Theorem 7.■

B.3. Time-Variant Backorder Cost and Holding Cost

In our discussion in section 3, we assume $h_\tau = 0$ and $p_\tau = p$ for simplicity of exposition. We relax these two assumptions to include explicitly time-variant backorder and holding cost. In fact, we only need to replace the expressions for some functions defined above. After it, a similar approach can be applied.

$V_0(\tau, y)$ is replaced by $\tilde{V}_0(\tau, y)$ defined below

$$\tilde{V}_0(\tau, y) = (p_\tau + h_\tau)\int_y^\infty (\xi - y) f_\tau(\xi)\,d\xi + h_\tau(y - \mu_\tau) + \int_0^\infty V_0(\tau - 1, y - \xi) f_\tau(\xi)\,d\xi$$

$D_1(\tau, y)$ is replaced by $\tilde{D}_1(\tau, y)$ defined below

$$\tilde{D}_1(\tau, y) = (p_\tau + h_\tau)\int_y^\infty (\xi - y) f_\tau(\xi)\,d\xi + h_\tau(y - \mu_\tau) + \int_0^\infty V_1(\tau - 1, y - \xi) f_\tau(\xi)\,d\xi$$

$\text{Đ}_1(\tau, y)$ is replaced by $\tilde{\text{Đ}}_1(\tau, y)$ defined below

$$\begin{aligned}\tilde{\text{Đ}}_1(\tau, y) &= \tilde{D}_1(\tau, y) + c_\tau y \\ &= \tilde{c}_\tau y + \tilde{p}_\tau \int_y^\infty (\xi - y) f_\tau(\xi)\,d\xi + \int_0^\infty V_1(\tau - 1, y - \xi) f_\tau(\xi)\,d\xi - h_\tau \mu_\tau\end{aligned}$$

where $\tilde{c}_\tau = c_\tau + h_\tau$ and $\tilde{p}_\tau = p_\tau + h_\tau$.

$\text{Đ}_k(\tau, y)$ is replaced by $\tilde{\text{Đ}}_k(\tau, y)$ defined below

$$\tilde{\text{Đ}}_k(\tau, y) = \tilde{c}_\tau y + \tilde{p}_\tau \int_y^\infty (\xi - y) f_\tau(\xi)\,d\xi + \int_0^\infty V_k(\tau - 1, y - \xi) f_\tau(\xi)\,d\xi - h_\tau \mu_\tau$$

Thus, these pairs of functions, $\tilde{D}_1(\tau, y)$ and $\text{Đ}_1(\tau, y)$, $\tilde{V}(T, x)$ and $V(T, x)$, $\tilde{\text{Đ}}_k(\tau, y)$ and $\text{Đ}_k(\tau, y)$, have similar expressions except a constant.

After replacing the functions defined above, all the conclusions in section 3 and section 4 are applicable provided that $\tilde{c}_\tau$ is nonincreasing in τ. (In case there is a unit salvage value v at the end of period T, then h_1 is h_1 minus v.) This completes our discussion for explicit inclusion of the time-variant holding and shortage cost.

B.4. Explicit Inclusion of a Fixed Order Cost

In this appendix, we show that, after explicitly incorporating a fixed order cost, K, into the purchase cost function, a time-varying (s,S) policy is still optimal.

We first discuss the one-order-opportunity case. Analogous to section3, it can be seen that $IP^*(\tau)$ is the optimal inventory position when an order is placed. Next, we show that it is not optimal to place an order when the inventory level is higher than a certain number. To do so, we define

$$ip^*(\tau) \hat{=} \arg\left\{0 \le y \le IP^*(\tau):\ c_\tau y + V_0(\tau, y) = K + c_\tau IP^*(\tau) + V_0\left(\tau, IP^*(\tau)\right)\right\}$$

Then, $ip^*(\tau)$ is well defined by the convexity of $c_\tau y + V_0(\tau, y)$ with respect to y. Now we have the following Lemma 3', instead of Lemma 3.

Lemma 3'. Assume we are at time $T-\tau+1$ and the inventory level is y. If $y \ge ip^*(\tau)$, then it is optimal not to place an order.

Proof Cost for the last τ periods if we do not place an order is $D_1(\tau, y)$. Cost if we place an order is

$$\ge \begin{cases} K + O_1(\tau, y, y) & \text{if } y \ge IP^*(\tau) \\ K + O_1\left(\tau, y, IP^*(\tau)\right) & \text{if } y < IP^*(\tau) \end{cases}$$
$$\ge O_1(\tau, y, y) = V_0(\tau, y)$$

Then the proof follows from the observation that $V_1(\tau-1, y) \le V_0(\tau-1, y)$.■

To summarize, there is a unique pair of $IP^*(\tau)$ and $ip^*(\tau)$ for each τ. If $y \ge ip^*(\tau)$, then it is optimal not to place an order. If $y < ip^*(\tau)$ then it <u>may be</u> optimal to place an order. If an order is placed then it is optimal to raise the inventory level to $IP^*(\tau)$ with the corresponding cost

$$O_1(\tau, y) = K + O_1\left(\tau, y, IP^*(\tau)\right) = K + c_\tau IP^*(\tau) + V_0\left(\tau, IP^*(\tau)\right) - c_\tau y.$$

Thus, we have

$$V_1(\tau,y)=\begin{cases}\min\{O_1(\tau,y),D_1(\tau,y)\} & \text{if } y<ip^*(\tau)\\ D_1(\tau,y) & \text{if } y\ge ip^*(\tau)\end{cases}.$$

We next focus on the case when $y<ip^*(\tau)$. To compare $O_1(\tau,y)$ and $D_1(\tau,y)$, it is sufficient to compare $K+c_\tau IP^*(\tau)+V_0(\tau,IP^*(\tau))$ and $Ð_1(\tau,y)$. Let

$$\Theta(\tau)=K+c(\tau)IP^*(\tau)+V_0(\tau,IP^*(\tau))$$

then it is optimal to place an order if $\Theta(\tau)<Ð_1(\tau,y)$.

The question that needs to be answered next is: Given that we are at time $T-\tau+1$ and inventory level $y<ip^*(\tau)$, for what values of y is it optimal to place an order? This can be answered similar to the case without fixed order cost if there is only one order opportunity. In particular, we can show that a time-varying (s,S) policy is optimal for each period.

We next discuss the multi-order-opportunity case. Similarly, we can define

$$ip_k^*(\tau)\triangleq\arg\left\{ip\le IP_k^*(\tau)\middle|O_k(\tau,0,ip)=K+O_k(\tau,0,IP_k^*(\tau))\right\}$$

Then, $ip_k^*(\tau)$ is well defined by the unimodality of $O_k(\tau,0,ip)$ with respect to ip. Now we have the following Lemma 6', instead of Lemma 6.

Lemma 6'. Assume we are at time $T-\tau+1$ and the inventory level is y. If $y\ge ip_k^*(\tau)$, then it is optimal not to place an order.

Proof Cost for the last τ periods if we do not place an order is $D_k(\tau,y)$. Cost if we place an order is not less than $O_k(\tau,y,y)$, analogous to Lemma 3. The proof follows from the observation that $V_k(\tau-1,y)\le V_{k-1}(\tau-1,y)$.■

In what follows, we only give the discussion for the case: $\tau=2(\le k)$ since all the other cases can be done by induction similar to the case without a fixed order cost.

Lemma 12.

1) $M_k(2,u)=\begin{cases} c_2-p+V_k'(1,y) & \text{if } y\le 0 \\ c_2+V_k'(1,y) & \text{if } y>0 \end{cases}$ changes sign once over $(-\infty,\infty)$ from "-" to "+".

2) $c_3+V_k'(1,y)$ changes sign at most once over $(0,\infty)$ from "-" to "+".

Proof 1) Since it can be shown, analogous to section 3, that $c_2-p+V_k'(1,y)\le 0$ for $y\le 0$, it is sufficient to show that $c_2+V_k'(1,y)$ changes sign at most once over $(0,\infty)$. Notice that

$$c_2+V_k'(1,y)=\begin{cases} c_2-c_1 & \text{if } y<y_k^*(1) \\ c_2-c_1+Đ_k'(1,y) & \text{if } y\ge y_k^*(1) \end{cases}$$

$c_2+V_k'(1,y)$ is negative if $y<y_k^*(1)$; for $y\ge y_k^*(1)$, $c_2+V_k'(1,y)$ increases to a positive value of c_2 as y approaches ∞ by the convexity of $Đ_k(1,y)$. Therefore $c_2+V_k'(1,y)$ changes sign at most once as y traverses from 0 to ∞. This leads to our conclusion.

2) It follows by a similar logic. ■

By Lemma 12, we can claim that $Đ_k'(2,y)$ changes sign once. Thus, $Đ_k(2,y)$ is unimodal, and there exists a unique $y=y_k^*(2)<IP_k^*(2)$ such that $Đ_k(2,y)$ intersects with $\Theta_k(2)$, i.e.,: an (s,S) policy with $s=y_k^*(2), S=IP_k^*(2)$ is optimal for $\tau=2$.

Similar discussion can be provided for other values of τ. This completes our discussion for explicit inclusion of a fixed ordering cost.

B.5. Explicit Inclusion of a Positve Delivery Leadtime

In this appendix, we show that, if there is a positive delivery leadtime L, it can be transformed into a case where the delivery leadtime is zero after some preliminary work (Clark and Scarf, 1960).

Let $\widetilde{V}_1(\tau,\tilde{y},x_1,\cdots,x_{L-1})$ denote the minimum expected cost over the last τ periods if there is 1 order opportunity left, we have $\tilde{y}$ units of inventory level at the beginning and $x_i\,(1\le i\le L-1)$ units of on-transit inventory that will be delivered at time $T-\tau+i+1$. Let $\widetilde{O}_1(\tau,\tilde{y},x_1,\cdots,x_{L-1})$ denote the minimum expected cost if an order is placed at time $T-\tau+1$. Let $\widetilde{D}_1(\tau,\tilde{y},x_1,\cdots,x_{L-1})$ denote the minimum expected cost if no order is placed. Then the ordering decision problem is formed as

$$\widetilde{V}_1(\tau,\tilde{y},x_1,\cdots,x_{L-1})= \min\left\{\widetilde{O}_1(\tau,\tilde{y},x_1,\cdots,x_{L-1}),\widetilde{D}_1(\tau,\tilde{y},x_1,\cdots,x_{L-1})\right\} \text{ for } L\le\tau\le T \qquad \text{... (P4.5)}$$

It is optimal not to place an order if $\widetilde{D}_1(\tau,\tilde{y},x_1,\cdots,x_{L-1})\le\widetilde{O}_1(\tau,\tilde{y},x_1,\cdots,x_{L-1})$.

For a given τ, to compare $\widetilde{D}_1(\tau,\tilde{y},x_1,\cdots,x_{L-1})$ and $\widetilde{O}_1(\tau,\tilde{y},x_1,\cdots,x_{L-1})$, we show that we only need to compare two one-variable functions $D_1(\tau,y)$ and $O_1(\tau,y)$ obtained through a transformation below, where $y=\tilde{y}+\sum_1^{L-1}x_i$ is the inventory position before the ordering decision. To do so, it is convenient for us to append the following definitions and notation.

Define $L_\tau(\tilde{y})=p\int_{\tilde{y}}^{\infty}(\xi-\tilde{y})f_\tau(\xi)d\xi$ for any $\tilde{y}$ in period $T-\tau+1$. Then $L_\tau(\tilde{y})$ is the expected backorder cost in period $T-\tau+1$.

Define

$$\mathcal{L}(\tau,\tilde{y},x_1,\cdots,x_{L-1})=L_{T-\tau+1}(\tilde{y})+\sum_{i=2}^{L}E\left[L_{T-\tau+i}\left(\tilde{y}+\sum_{j=1}^{i-1}x_j-\xi_{\tau,i-1}\right)\right]$$

$\xi_{\tau,i}$ is the demand in periods $T-\tau+1$ through $T-\tau+i$. $\mathcal{L}(\tau,\tilde{y},x_1,\cdots,x_{L-1})$ is the expected backorder cost in periods $T-\tau+1$ through $T-\tau+L$.

Define

$$\widetilde{V}_0\left(\tau,\tilde{y},x_1,\cdots,x_{L-1},IP-\tilde{y}-\sum_1^{L-1}x_i\right)$$
$$=L_\tau(\tilde{y})+\int_0^\infty \widetilde{V}_0\left(\tau-1,\tilde{y}+x_1-\xi,x_2,\cdots,IP-\tilde{y}-\sum_1^{L-1}x_i,0\right)f_\tau(\xi)\,d\xi$$

$\widetilde{V}_0\left(\tau,\tilde{y},x_1,\cdots,x_{L-1},IP-\tilde{y}-\sum_1^{L-1}x_i\right)$ is the expected cost over the last τ periods. It is easy to see that

$$\widetilde{V}_0\left(\tau,\tilde{y},x_1,\cdots,x_{L-1},IP-\tilde{y}-\sum_1^{L-1}x_i\right)$$
$$=L_\tau(\tilde{y})+\int_0^\infty \widetilde{V}_0\left(\tau-1,\tilde{y}+x_1-\xi,x_2,\cdots,IP-\tilde{y}-\sum_1^{L-1}x_i,0\right)f_\tau(\xi)\,d\xi$$
$$=\mathcal{L}(\tau,\tilde{y},x_1,\cdots,x_{L-1})+E\left[\widetilde{V}_0\left(\tau-L,IP-\xi_{\tau,L},0,\cdots,0,0\right)\right]$$
$$=\mathcal{L}(\tau,\tilde{y},x_1,\cdots,x_{L-1})+V_0(\tau,IP)$$

where

$$V_0(\tau,IP)=E\left[\widetilde{V}_0\left(\tau-L,IP-\xi_{\tau,L},0,\cdots,0,0\right)\right]$$
$$=E\left[L_{\tau-L}\left(IP-\xi_{\tau,L}\right)\right]+E\left[\int_0^\infty \widetilde{V}_0\left(\tau-L-1,IP-\xi_{\tau,L}-\xi,0,\cdots,0,0\right)f_{\tau-L}(\xi)\,d\xi\right]$$
$$=p\int_{IP}^\infty\left(\xi_{\tau,L+1}-IP\right)f_{\tau,L+1}(\xi)\,d\xi+\int_0^\infty V_0\left(\tau-1,IP-\xi_{\tau,L+1}\right)f_{\tau,L+1}(\xi)\,d\xi$$

Therefore, $V_0(\tau,IP)$ is the expected cost for the last $\tau-L$ periods, viewed at time $T-\tau+1$.

Define

$$\widetilde{O}_1\left(\tau,\tilde{y},x_1,\cdots,x_{L-1},IP\right)=c(\tau)(IP-y)+\widetilde{V}_0\left(\tau,\tilde{y},x_1,\cdots,x_{L-1},IP-y\right)$$

Then,

$$\widetilde{O}_1\left(\tau,\tilde{y},x_1,\cdots,x_{L-1},IP\right)=c(\tau)(IP-y)+\mathcal{L}(\tau,\tilde{y},x_1,\cdots,x_{L-1})+V_0(\tau,IP)$$
$$=\mathcal{L}(\tau,\tilde{y},x_1,\cdots,x_{L-1})+O_1(\tau,y,IP)$$

where $O_1(\tau, y, IP) = c(\tau)(IP - y) + V_0(\tau, IP)$. This leads to

$$\begin{aligned}\widetilde{O}_1(\tau, \tilde{y}, x_1, \cdots, x_{L-1}) &= \min_{IP \geq y}\left\{\widetilde{O}_1(\tau, \tilde{y}, x_1, \cdots, x_{L-1}, IP)\right\} \\ &= O_1(\tau, y) + \mathcal{L}(\tau, \tilde{y}, x_1, \cdots, x_{L-1})\end{aligned} \quad \ldots (20)$$

where $O_1(\tau, y) = \min_{IP \geq y}\{O_1(\tau, y, IP)\}$ is the minimum expected cost for the last $\tau - L$ periods if we place an order at time $T - \tau + 1$, viewed at time $T - \tau + 1$.

Let $V_1(\tau, y)$ be the minimum expected cost for the last $\tau - L$ periods, viewed at time $T - \tau + 1$. Then, we can get

$$\begin{aligned}&\widetilde{D}_1(\tau, \tilde{y}, x_1, \cdots, x_{L-1}) \\ &= L_\tau(\tilde{y}) + \int_0^\infty \widetilde{V}_1(\tau - 1, \tilde{y} + x_1 - \xi, x_2, \cdots, x_{L-1}, 0) f_\tau(\xi) d\xi \\ &= L_\tau(\tilde{y}) + \int_0^\infty \left(\mathcal{L}(\tau - 1, \tilde{y} + x_1 - \xi, x_2, \cdots, x_{L-1}, 0) + V_1(\tau - 1, y - \xi)\right) f_\tau(\xi) d\xi \\ &= \mathcal{L}(\tau, \tilde{y}, x_1, \cdots, x_{L-1}) + D_1(\tau, y)\end{aligned} \quad \ldots (21)$$

where $D_1(\tau, y) = p\int_{IP}^\infty (\xi_{\tau,L+1} - IP) f_{\tau,L+1}(\xi) d\xi + \int_0^\infty V_1(\tau - 1, y - \xi) f_\tau(\xi) d\xi$ is the minimum expected cost for the last $\tau - L$ periods if we do not place an order at time $T - \tau + 1$, viewed at time $T - \tau + 1$.

By (20), (21) and (P4.5), we have

$$V_1(\tau, y) = \widetilde{V}_1(\tau, \tilde{y}, x_1, \cdots, x_{L-1}) - \mathcal{L}(\tau, \tilde{y}, x_1, \cdots, x_{L-1}) = \min\{O_1(\tau, y), D_1(\tau, y)\}$$

For the case we have k order opportunities left for the last τ periods, we define

$$\begin{aligned}\widetilde{O}_k(\tau, \tilde{y}, x_1, \cdots, x_{L-1}, IP) &= c(\tau)(IP - y) + \mathcal{L}(\tau, \tilde{y}, x_1, \cdots, x_{L-1}) + V_{k-1}(\tau, IP) \\ &= \mathcal{L}(\tau, \tilde{y}, x_1, \cdots, x_{L-1}) + O_k(\tau, y, IP)\end{aligned}$$

where $O_k(\tau, y, IP) = c(\tau)(IP - y) + V_{k-1}(\tau, IP)$. Therefore, we can get

$$\begin{aligned}\widetilde{O}_k(\tau, \tilde{y}, x_1, \cdots, x_{L-1}) &= \min_{IP \geq y}\left\{\widetilde{O}_k(\tau, \tilde{y}, x_1, \cdots, x_{L-1}, IP)\right\} \\ &= O_k(\tau, y) + \mathcal{L}(\tau, \tilde{y}, x_1, \cdots, x_{L-1})\end{aligned} \quad \ldots (22)$$

where $O_k(\tau, y) = \min_{IP \geq y}\{O_k(\tau, y, IP)\}$. Let $V_k(\tau, y)$ be the minimum expected cost for the last $\tau - L$ periods, viewed at time $T - \tau + 1$. Then, we can get

$$\begin{aligned}
&\widetilde{D_k}\left(\tau,\tilde{y},x_1,\cdots,x_{L-1}\right)\\
&=L_\tau\left(\tilde{y}\right)+\int_0^\infty \widetilde{V_k}\left(\tau-1,\tilde{y}+x_1-\xi,x_2,\cdots,x_{L-1},0\right)f_\tau\left(\xi\right)d\xi\\
&=L_\tau\left(\tilde{y}\right)+\int_0^\infty\left(\mathcal{L}\left(\tau-1,\tilde{y}+x_1-\xi,x_2,\cdots,x_{L-1},0\right)+V_k\left(\tau-1,y-\xi\right)\right)f_\tau\left(\xi\right)d\xi\\
&=\mathcal{L}\left(\tau,\tilde{y},x_1,\cdots,x_{L-1}\right)+D_k\left(\tau,y\right)
\end{aligned} \quad \ldots (23)$$

where $D_k\left(\tau,y\right)=p\int_{IP}^\infty\left(\xi_{\tau,L+1}-IP\right)f_{\tau,L+1}\left(\xi_{\tau,L+1}\right)d\xi_{\tau,L+1}+\int_0^\infty V_k\left(\tau-1,y-\xi\right)f_\tau\left(\xi\right)d\xi$.

Similarly, we have

$$V_k\left(\tau,y\right)=\widetilde{V}_k\left(\tau,\tilde{y},x_1,\cdots,x_{L-1}\right)-\mathcal{L}\left(\tau,\tilde{y},x_1,\cdots,x_{L-1}\right)=\min\left\{O_k\left(\tau,y\right),D_k\left(\tau,y\right)\right\}.$$

The discussion above shows that, to study the case with positive leadtime, we can construct a transformation so that we only need to study the after-transformation case, making use of the techniques for the zero-leadtime case.

Appendix C. Omitted Proofs in Chapter 4

C.1. Proof of Theorem 8.

1) Recall that

$$V_t(\underline{z},z,IP)=c_{t,e}(z-\underline{z})+c_{t,n}(IP-z)+L_t(z,IP)+\int_0^{\infty}R_t(IP-\xi_{\lambda^-})\tilde{f}_{t,\lambda^-}(\xi_{\lambda^-})d\xi_{\lambda^-}$$

Differentiating it with respect to IP yields

$$\frac{\partial V_t(\underline{z},z,IP)}{\partial IP}=$$

$$c_{t,n}+\frac{\partial L_t(z,IP)}{\partial IP}+\int_{IP}^{\infty}\frac{\partial R_t(IP-\xi_{\lambda^-})}{\partial IP}\tilde{f}_{t,\lambda^-}(\xi_{\lambda^-})d\xi_{\lambda^-}+\int_0^{IP}\frac{\partial R_t(IP-\xi_{\lambda^-})}{\partial IP}\tilde{f}_{t,\lambda^-}(\xi_{\lambda^-})d\xi_{\lambda^-}$$

$$=c_{t,n}+\int_{IP}^{\infty}\left(p_{t,\lambda^-}+\frac{\partial R_t(IP-\xi_{\lambda^-})}{\partial IP}\right)\tilde{f}_{t,\lambda^-}(\xi_{\lambda^-})d\xi_{\lambda^-}+\int_0^{IP}\frac{\partial R_t(IP-\xi_{\lambda^-})}{\partial IP}\tilde{f}_{t,\lambda^-}(\xi_{\lambda^-})d\xi_{\lambda^-}$$

$$=\int_{-\infty}^{\infty}M_t(u)\tilde{f}_{t,\lambda^-}(IP-u)du$$

where

$$M_t(u)=\begin{cases} c_{t,n}+p_{t,\lambda^-}+\dfrac{dR_t(u)}{du} & \text{If } u<0 \\ c_{t,n}+\dfrac{dR_t(u)}{du} & \text{If } u\geq 0 \end{cases}$$

It can be seen that $M_t(u)$ changes sign once over $(-\infty,\infty)$. In fact, for $u<0$, we have

$$\frac{dR_t(u)}{du}=-p_t+\int_0^{\infty}V_{t-1}'(u-\xi_{\lambda^+})\tilde{f}_{t,\lambda^+}(\xi_{\lambda^+})d\xi_{\lambda^+}$$
$$=-p_t-c_{t-1,e}$$

The second equality is true by the induction assumption that $V_{t+1}'(\underline{z})=-c_{t-1,e}$ for $\underline{z}\leq 0$. It implies that $M_t(u)=c_{t,n}+p_{t,\lambda^-}+\frac{dR_t(u)}{du}=c_{t,n}+p_{t,\lambda^-}-p_t-c_{t-1,e}<0$ by the assumption that no-backordering motivation exists. For $u\geq 0$, $c_{t,n}+\frac{dR_t(u)}{du}$

changes sign at most once by the convexity of $R_t(u)$. Therefore, $M_t(u)$ changes sign once as u traverses over $(-\infty,\infty)$.

As a result, $\frac{\partial V_t(\underline{z},z,IP)}{\partial IP}$ changes sign once over $(-\infty,\infty)$ with respect to IP. Therefore, $V_t(\underline{z},z,IP)$ is unimodal in IP and the minimum is achieved at $IP_t^*(>0)$.

2) The convexity of $V_t(\underline{z},z,IP)$ with respect to z is implied by the fact it is a combination of two convex functions with respect to z. It is easy to see that its first order derivative is

$$\frac{\partial V_t(\underline{z},z,IP)}{\partial z} = c_{t,e} - c_{t,n} + h_{t,\lambda^-} - \left(p_{t,\lambda^-} + h_{t,\lambda}\right)\int_z^\infty \tilde{f}_{t,\lambda}\left(\xi_\lambda\right)d\xi_\lambda .$$

Setting the derivative equal to 0 implies $\int_{z_t^*}^\infty \tilde{f}_{t,\lambda^-}\left(\xi_{\lambda^-}\right)d\xi_{\lambda^-} = \frac{c_{t,e} - c_{t,n} + h_{t,\lambda^-}}{p_{t,\lambda^-} + h_{t,\lambda^-}} < 1$, and therefore $z_t^* > 0$.■

C.2. Proof of Theorem 9.

Recall that $V_t(\underline{z})$ has an expression

$$V_t(\underline{z}) = \begin{cases} V_t\left(\underline{z}, z_t^*, IP_t^*\right) & \text{If } \underline{z} \le z_t^* \\ V_t\left(\underline{z}, \underline{z}, IP_t^*\right) & \text{If } z_t^* < \underline{z} \le IP_t^* \\ V_t\left(\underline{z}, \underline{z}, \underline{z}\right) & \text{If } \underline{z} > IP_t^* \end{cases}.$$

Taking derivatives for $V_t(\underline{z})$ with respect to $\underline{z}$, we have

$$\frac{dV_t(\underline{z})}{d\underline{z}} = \begin{cases} -c_{t,e} & \text{If } \underline{z} \le z_t^* \\ -c_{t,n} + \frac{\partial L_t\left(\underline{z}, IP_t^*\right)}{\partial \underline{z}} & \text{If } z_t^* < \underline{z} \le IP_t^* \\ -c_{t,n} + \left.\left(\frac{\partial L_t(\underline{z},IP)}{\partial \underline{z}} + \frac{\partial V_t(\underline{z},\underline{z},IP)}{\partial IP}\right)\right|_{IP=\underline{z}} & \text{If } \underline{z} > IP_t^* \end{cases}$$

$$\frac{d^2V_t(\underline{z})}{d\underline{z}^2}=\begin{cases}0 & \text{If } \underline{z}\le z_t^* \\ \dfrac{\partial^2 L_t(\underline{z},IP_t^*)}{\partial \underline{z}^2} & \text{If } z_t^*<\underline{z}\le IP_t^* \\ \dfrac{d^2V_t(\underline{z},\underline{z},\underline{z})}{d\underline{z}^2} & \text{If } \underline{z}>IP_t^*\end{cases}$$

We can claim $\frac{d^2V_t(\underline{z},\underline{z},\underline{z})}{d\underline{z}^2}\ge 0$ by the convexity of $V_t(\underline{z},\underline{z},\underline{z})$ and $L_t(\underline{z},IP_t^*)$ with respect to $\underline{z}$. The convexity of $V_t(\underline{z},\underline{z},\underline{z})$ is implied by the following expression:

$$V_t(\underline{z},\underline{z},\underline{z})=L_t(\underline{z},\underline{z})+\int_0^\infty R_t(\underline{z}-\xi_{\lambda^-})\tilde{f}_{t,\lambda^-}(\xi_{\lambda^-})d\xi_{\lambda^-},$$

where $L_t(\underline{z},\underline{z})=h_{t,\lambda^-}\int_0^{z}(z-\xi_{\lambda^-})\tilde{f}_{t,\lambda^-}(\xi_{\lambda^-})d\xi_{\lambda^-}$ is convex in $\underline{z}(>0)$ and $R_t(y)$ is convex in y. We are now ready to claim that $V_t(\underline{z})$ is convex in $\underline{z}$ since $c_{t,e}+\frac{\partial L_t(\underline{z},IP_t^*)}{\partial \underline{z}}\ge c_{t,n}$ for $z_t^*<\underline{z}$ and $\left.\frac{\partial V_t(\underline{z},\underline{z},IP)}{\partial IP}\right|_{IP=\underline{z}}\ge 0$ for $\underline{z}>IP_t^*$. It is seen from above that $V_t'(\underline{z})=-c_{t,e}$ for $\underline{z}\le 0$. ■

C.3. Proof of Theorem 10.

1) Differentiating $V_1(\underline{z},z,IP)$ with respect to IP yields

$$\frac{\partial V_1(\underline{z},z,IP)}{\partial IP}=\int_{-\infty}^{\infty}M_1(u)\tilde{f}_{1,\lambda^-}(IP-u)du$$

where

$$M_1(u)=\begin{cases}c_{1,n}+p_{1,\lambda^-}-p_1 & \text{If } u<0 \quad \left(\text{since } \dfrac{dR_1(u)}{du}=-p_1\right) \\ c_{1,n}+\dfrac{dR_1(u)}{du} & \text{If } u\ge 0\end{cases}$$

It can be seen that $M_1(u)$ changes sign once over $(-\infty,\infty)$. As a result, $\frac{\partial V_1(\underline{z},z,IP)}{\partial IP}$ changes sign once over $(-\infty,\infty)$ with respect to IP. Therefore, $V_1(\underline{z},z,IP)$ is unimodal in IP and the minimum is achieved at $IP_1^*(>0)$.

2) Similar to the proof for part 2 of Theorem 8.

3) Similar to the proof for Theorem 9. ■

C.4. Proof of Proposition 1.

It is seen from Theorem 9 that z_t^* satisfies $\frac{\partial V_t(\underline{z},z,IP)}{\partial z}=0$, i.e., $\int_{z_t^*}^{\infty}\tilde{f}_{t,\lambda^-}(\xi_{\lambda^-})d\xi_{\lambda^-}=\frac{c_{t,e}-c_{t,n}+h_{t,\lambda^-}}{p_{t,\lambda^-}+h_{t,\lambda^-}}$. Then the proof follows from basic probability properties. ■

C.5. Proof of Proposition 2.

With a little algebra, we can have

$$\begin{aligned}\frac{\partial V_t(\underline{z},z,IP)}{\partial IP}&=c_{t,n}+\left(p_{t,\lambda^-}-p_t\right)\int_{IP}^{\infty}\tilde{f}_{t,\lambda^-}(\xi_{\lambda^-})d\xi_{\lambda^-}-p_{t,\lambda^+}\int_0^{IP}\int_{IP-\xi_{\lambda^-}}^{\infty}\tilde{f}_{t,\lambda^+}(\xi_{\lambda^+})\tilde{f}_{t,\lambda^-}(\xi_{\lambda^-})d\xi_{\lambda^+}d\xi_{\lambda^-}\\&\quad+h_{t,\lambda^+}\int_0^{IP}\int_0^{IP-\xi_{\lambda^-}}\tilde{f}_{t,\lambda^+}(\xi_{\lambda^+})\tilde{f}_{t,\lambda^-}(\xi_{\lambda^-})d\xi_{\lambda^+}d\xi_{\lambda^-}\\&\quad+\int_0^{\infty}\int_0^{\infty}V_{t-1}'(IP-\xi_{\lambda^+}-\xi_{\lambda^-})\tilde{f}_{t,\lambda^+}(\xi_{\lambda^+})\tilde{f}_{t,\lambda^-}(\xi_{\lambda^-})d\xi_{\lambda^+}d\xi_{\lambda^-}\end{aligned}$$

Thus, $\frac{\partial V_t(\underline{z},z,IP)}{\partial IP}$ increases as one of $c_{t,n}$, p_{t,λ^-} and h_{t,λ^+} increases; And $\frac{\partial V_t(\underline{z},z,IP)}{\partial IP}$ decreases as one of p_{t,λ^+} and p_t increase. These observations lead to Proposition 2 by the unimodality of $V_t(\underline{z},z,IP)$. ■

C.6. Proof of Theorem 11.

1) If $c_{t^\circ,e} - c_{t^\circ,n} + h_{t^\circ,\lambda^-} \le 0$, then $\dfrac{\partial V_{t^\circ}(\underline{z}, z, IP)}{\partial z} < 0$ for any $z > 0$. By the convexity of $V_{t^\circ}(\underline{z}, z, IP)$ with respect to z, we see that the optimal value of z, z_t^*, would be as large as possible. However, it is limited from above by $IP_{t^\circ}^*$. Thus $z_{t^\circ}^* = IP_{t^\circ}^*$. The proof for the convexity is similar to the proof for Theorems 8 and 9.

2) If $c_{t^\circ,e} \ge c_{t^\circ,n} + p_{t^\circ,\lambda^-}$, then $\dfrac{\partial V_{t^\circ}(\underline{z}, z, IP)}{\partial z} > 0$ for any $z > 0$. Therefore, $z_{t^\circ}^*$ would be as small as possible. However, it is limited from below by $\underline{z}$. Thus $z_{t^\circ}^* = \underline{z}$. From Theorem 8, $IP_{t^\circ}^*$ is uniquely determined by $\dfrac{\partial V_{t^\circ}(\underline{z}, \underline{z}, IP)}{\partial IP} = 0$. The proof for the convexity and unimodality is similar to the proof for Theorems 8 and 9.■

C.7. Discussion for the Shortage Assumption

Throughout our discussion above, we assumed that shortages in a period are backordered with penalty feature. In this appendix, we discuss its opposite cases: case I assumes that, in every period, shortages unmet from the slow delivery are lost; and case II assumes that, in every period, shortages are all lost or backordered without penalty feature. Although case I still requires demand functions to be PF_2, case II does not require so. We follow similar notation with a little adjustment.

C.7.1. Case I

In this subsection, we assume that the demand unmet from the slow delivery is lost in each period. p_{t,λ^-} represents unit backorder cost in the first interval; p_{t,λ^+} represents the unit lost-sale cost in period $T-t+1$. We assume that $p_{t,\lambda^+} + h_{t,\lambda^+} - c_{t-1,e} \ge 0$, $c_{t,n} + p_{t,\lambda^-} > c_{t,e}$, and $p_{t,\lambda^+} > \max\{c_{t,n}, c_{t,e}, c_{t-1,e}\}$. We adjust the expressions for $R_t(y)$, $L_t(z, IP)$ and $V_t(\underline{z}, z, IP)$ as follows

$$R_t(y) = p_{t,\lambda^+} E\left[(\xi_{\lambda^+} - y)^+\right] + h_{t,\lambda^+} E\left[(y - \xi_{\lambda^+})^+\right] + E\left[V_{t-1}\left((y - \xi_{\lambda^+})^+\right)\right]$$

$$L_t(z,IP)=p_{t,\lambda^-}E\left[\min\left\{\left(\xi_{\lambda^-}-z\right)^+,\left(IP-z\right)\right\}\right]+h_{t,\lambda^-}E\left[\left(z-\xi_{\lambda^-}\right)^+\right]$$

$$V_t(\underline{z},z,IP)=c_{t,e}(z-\underline{z})+c_{t,n}(IP-z)+L_t(z,IP)+\int_0^\infty R_t\left(IP-\xi_{\lambda^-}\right)\tilde{f}_{t,\lambda^-}\left(\xi_{\lambda^-}\right)d\xi_{\lambda^-}$$

It is easy to see that $L_t(z,IP)$ is separable, convex in z and concave in IP, and that $R_t(y)$ is convex in y. Using an approach similar to that in Section 3, we can show that $V_t(\underline{z},z,IP)$ is unimodal in IP, and that $V_t(\underline{z},z,IP)$ is convex in z. Corresponding results can be derived.

C.7.2. Case II

For the cases where shortages are all lost or backordered without penalty feature, we do not require that demand density functions fall in PF_2. We first discuss the case for lost-sales. Then we discuss the case for backordering.

In the case for all lost-sales, IP means the quantity for the slow order, instead of the inventory position after the purchase decision at the beginning of each period. p_{t,λ^-} and p_{t,λ^+} represent, respectively, unit lost-sale costs in the first and second interval of period $T-t+1$. The following assumptions are used: 1) $p_{t,\lambda^+}+h_{t,\lambda^+}-c_{t-1,e}\geq 0$; 2) $p_{t,\lambda^-}+h_{t,\lambda^-}-p_{t,\lambda^+}\geq 0$. We adjust the expressions for $R_t(y)$, $L_t(z,IP)$ and $V_t(\underline{z},z,IP)$ as follows

$$R_t(y)=p_{t,\lambda^+}E\left[\left(\xi_{\lambda^+}-y\right)^+\right]+h_{t,\lambda^+}E\left[\left(y-\xi_{\lambda^+}\right)^+\right]+E\left[V_{t-1}\left(\left(y-\xi_{\lambda^+}\right)^+\right)\right]$$

$$L_t(z,IP)=p_{t,\lambda^-}E\left[\left(\xi_{\lambda^-}-z\right)^+\right]+h_{t,\lambda^-}E\left[\left(z-\xi_{\lambda^-}\right)^+\right]$$

$$V_t(\underline{z},z,IP)=c_{t,e}(z-\underline{z})+c_{t,n}IP+L_t(z,IP)+\int_0^\infty R_t\left(IP+\left(z-\xi_{\lambda^-}\right)^+\right)\tilde{f}_{t,\lambda^-}\left(\xi_{\lambda^-}\right)d\xi_{\lambda^-}$$

We are now ready to present the decision problem in period $T-t+1$ below.

$$V_t(\underline{z})=\min_{z\geq\underline{z},IP\geq 0}\left\{V_t(\underline{z},z,IP)\right\} \quad \text{... (P5.2)}$$

That is, the decision is to choose the optimal inventory level z_t^* and the optimal order quantity IP_t^*. $V_t(\underline{z})$ is therefore the minimum expected cost for the last t

periods with a beginning inventory level $\underline{z}$ before the order decision. We assume $V_0(\underline{z}) \equiv 0$.

We solve (P5.2) in a recursive way. In particular, we show that 1) $V_t(\underline{z})$ is convex and $V_t'(0) = -c_{t,e}$, provided that $V_{t-1}(\underline{z})$ is convex and $V_{t-1}'(0) = -c_{t-1,e}$; and 2) $V_1(\underline{z})$ is convex and $V_1'(0) = -c_{1,e}$.

To do so, we first study the properties of the functions $R_t(y)$ and $L_t(z, IP)$. In particular, we show that 1) $R_t(IP)$ is convex in IP; 2) $L_t(z, IP)$ is convex in z.

The convexity of $R_t(y)$ can be seen from the derivatives below:

$$\frac{\partial R_t(y)}{\partial y} = -\left(p_{t,\lambda^+} + h_{t,\lambda^+}\right) \int_y^\infty \tilde{f}_{t,\lambda^+}\left(\xi_{\lambda^+}\right) d\xi_{L+1,T} + \int_0^y V_{t-1}'\left(y - \xi_{\lambda^+}\right) \tilde{f}_{t,\lambda^+}\left(\xi_{\lambda^+}\right) + h_{t,\lambda^+}$$

$$\frac{\partial^2 R_t(y)}{\partial y^2} = \left(p_{t,\lambda^+} + h_{t,\lambda^+} - c_{t-1,e}\right) \tilde{f}_{t,\lambda^+}(y) + \int_0^{IP} V_{t-1}''\left(y - \xi_{\lambda^+}\right) \tilde{f}_{t,\lambda^+}\left(\xi_{\lambda^+}\right)$$

where $\dfrac{\partial^2 R_t(y)}{\partial IP^2} \geq 0$ is because both terms are non-negative. In addition, we have $R_t'(0) = -p_{t,\lambda^+}$.

$L_t(z, IP)$ is free of IP. The convexity of $L_t(z, IP)$ with respect to z is seen from the fact $L_t(z, IP)$ is effectively the sum of the newsvendor-type expected shortage and holding costs for a beginning inventory z.

We are now ready to derive the properties for $V_t(\underline{z}, z, IP)$. We form it in the following theorem.

Theorem 12.

1) $V_t(\underline{z}, z, IP)$ is convex in IP for given $z(\geq \underline{z})$, and there is a unique $IP_t^*(z)$ minimizing $V_t(\underline{z}, z, IP)$ with respect to IP over $[0, \infty)$.

2) $V_t\left(\underline{z},z,IP_t^*(z)\right)$ is convex in z, and there is a unique $z_t^*(>0)$ satisfying

$$\frac{\partial V_t\left(\underline{z},z,IP_t^*(z)\right)}{\partial z}=c_{t,e}+h_{t,\lambda^-}-\left(p_{t,\lambda^-}+h_{t,\lambda^-}\right)\int_z^\infty \tilde{f}_{t,\lambda^-}\left(\xi_{\lambda^-}\right)d\xi_{\lambda^-}$$
$$+\int_0^z R_t'\left(IP_t^*(z)+z-\xi_{\lambda^-}\right)\tilde{f}_{t,\lambda^-}\left(\xi_{\lambda^-}\right)d\xi_{\lambda^-}=0$$

Proof. 1) With a little algebra, differentiating $V_t\left(\underline{z},z,IP\right)$ with respect to IP yields

$$\frac{\partial V_t\left(\underline{z},z,IP\right)}{\partial IP}=c_{t,n}+\int_0^z R_t'\left(IP+z-\xi_{\lambda^-}\right)\tilde{f}_{t,\lambda}\left(\xi_\lambda\right)d\xi_\lambda+\int_z^\infty R_t'\left(IP\right)\tilde{f}_{t,\lambda}\left(\xi_\lambda\right)d\xi_\lambda$$

$$\frac{\partial^2 V_t\left(\underline{z},z,IP\right)}{\partial IP^2}=\int_0^z R_t''\left(IP+z-\xi_{\lambda^-}\right)\tilde{f}_{t,\lambda^-}\left(\xi_{\lambda^-}\right)d\xi_{\lambda^-}+\int_z^\infty R_t''\left(IP\right)\tilde{f}_{t,\lambda^-}\left(\xi_{\lambda^-}\right)d\xi_{\lambda^-}\geq 0.$$

The expressions above imply that $V_t\left(\underline{z},z,IP\right)$ is convex in IP for given z by the convexity of $R_t(y)$. Therefore $IP_t^*(z)$ is unique for a given $z(\geq \underline{z})$, and $IP_t^*(z)$ is continuous in z. In addition, we have

$$\frac{dIP_t^*(z)}{dz}=-\left.\frac{\int_0^z R_t''\left(IP+z-\xi_{\lambda^-}\right)\tilde{f}_{t,\lambda^-}\left(\xi_{\lambda^-}\right)d\xi_{\lambda^-}}{\int_0^z R_t''\left(IP+z-\xi_{\lambda^-}\right)\tilde{f}_{t,\lambda^-}\left(\xi_{\lambda^-}\right)d\xi_{\lambda^-}+\int_z^\infty R_t''\left(IP\right)\tilde{f}_{t,\lambda^-}\left(\xi_{\lambda^-}\right)d\xi_{\lambda^-}}\right|_{IP=IP_t^*(z)}.$$

2) Recall the expressions for $V_t\left(\underline{z},z,IP_t^*(z)\right)$, $L_t\left(z,IP\right)$ and $R_t(y)$. With a little algebra, we get

$$\frac{\partial V_t\left(\underline{z},z,IP_t^*(z)\right)}{\partial z}=c_{t,e}+h_{t,\lambda^-}-\left(p_{t,\lambda^-}+h_{t,\lambda^-}\right)\int_z^\infty \tilde{f}_{t,\lambda^-}\left(\xi_{\lambda^-}\right)d\xi_{\lambda^-}+\int_0^z R_t'\left(IP_t^*+z-\xi_{\lambda^-}\right)\tilde{f}_{t,\lambda^-}\left(\xi_{\lambda^-}\right)d\xi_\lambda$$

$$\frac{\partial^2 V_t\left(\underline{z},z,IP_t^*(z)\right)}{\partial z^2}=\left(p_{t,\lambda^-}+h_{t,\lambda^-}+R_t'\left(IP_t^*(z)\right)\right)\tilde{f}_{t,\lambda^-}(z)$$
$$+\int_0^z\left(1+\frac{dIP_t^*(z)}{dz}\right)R_t''\left(IP_t^*(z)+z-\xi_{\lambda^-}\right)\tilde{f}_{t,\lambda^-}\left(\xi_{\lambda^-}\right)d\xi_{\lambda^-}$$
$$\geq 0 \quad \left(\because\ R_t'\left(IP_t^*\right)\geq -p_{t,\lambda^+}\right)$$

Therefore $\frac{\partial^2 V_t(\underline{z},z,IP_t^*(z))}{\partial z^2} \geq 0$ implies $V_t(\underline{z},z,IP_t^*(z))$ is convex in z. It is easy to see that $z_t^* > 0$ from $\left.\frac{\partial V_t(\underline{z},z,IP_t^*(z))}{\partial z}\right|_{z=0} < 0$. ■

Then the optimal order rule at time t is: 1) to order $IP_t^*(z_t^*)$ units for the slow delivery and order $z_t^* - \underline{z}$ units for the fast delivery if $\underline{z} \leq z_t^*$; 2) to order $IP_t^*(\underline{z})$ units for the slow delivery if $\underline{z} > z_t^*$. Then by the definition of $V_t(\underline{z})$, we have

$$V_t(\underline{z}) = \begin{cases} V_t(\underline{z},z_t^*,IP_t^*(z_t^*)) & \text{If } \underline{z} < z_t^* \\ V_t(\underline{z},\underline{z},IP_t^*(\underline{z})) & \text{If } \underline{z} \geq z_t^* \end{cases}.$$

We next show that $V_t(\underline{z})$ is convex in $\underline{z}$. We form it in the following theorem.

Theorem 13. $V_t(\underline{z})$ is convex in $\underline{z}$ and $V_t'(0) = -c_{t,e}$.

Proof. Taking derivatives for $V_t(\underline{z})$ yields

$$\frac{dV_t(\underline{z})}{d\underline{z}} = \begin{cases} -c_{t,e} & \text{If } \underline{z} < z_t^* \\ -c_{t,e} + \left.\frac{\partial V_t(\underline{z},z,IP_t^*(z))}{\partial z}\right|_{z=\underline{z}} & \text{If } \underline{z} \geq z_t^* \end{cases}$$

Since $V_t(\underline{z},z,IP_t^*(z))$ is convex in z and its minimum is obtained at $z = z_t^*$, $\frac{\partial V_t(\underline{z},z,IP_t^*(z))}{\partial z} \geq 0$ for $z \geq z_t^*$. Thus $\left.\frac{dV_t(\underline{z})}{d\underline{z}}\right|_{\underline{z}=z_1} \leq \left.\frac{dV_t(\underline{z})}{d\underline{z}}\right|_{\underline{z}=z_2}$ for any $z_1 < z_t^*, z_2 \geq z_t^*$. Thus, we can claim that $V_t(\underline{z})$ is convex in $\underline{z}$, taking into account that $\frac{d^2 V_t(\underline{z})}{d\underline{z}^2} \geq 0$. It is obvious that $V_t'(0) = -c_{t,e}$. ■

Following the above logic, we can show that $V_1(\underline{z})$ is convex (since $V_0(\underline{z}) \equiv 0$) and $V_1'(0) = -c_{1,e}$. This completes the induction.

The static case of Milner and Kouvelis (2002) discussed essentially a one-period variant of our problem. The partial dynamic case of Milner and Kouvelis (2002) discussed essentially a two-period variant of our problem where the first period has only the fast-delivery mode and the second period has only the slow-delivery mode.

In the case when all shortages are backordered, IP represents the inventory position after the purchase decision. p_{t,λ^-} and p_{t,λ^+} represent, respectively, unit backorder costs in the first and second interval of period $T-t+1$. We assume that $c_{t,e} < c_{t,n} + p_{t,\lambda^-}$. We adjust the expressions for $R_t(y)$, $L_t(z, IP)$ and $V_t(\underline{z}, z, IP)$ as follows:

$$R_t(y) = p_{t,\lambda^+} E\left[\left(\xi_{\lambda^+} - y\right)^+\right] + h_{t,\lambda^+} E\left[\left(y - \xi_{\lambda^+}\right)^+\right] + E\left[V_{t-1}\left(y - \xi_{\lambda^+}\right)\right]$$

$$L_t(z, IP) = p_{t,\lambda^-} E\left[\left(\xi_{\lambda^-} - z\right)^+\right] + h_{t,\lambda^-} E\left[\left(z - \xi_{\lambda^-}\right)^+\right]$$

$$V_t(\underline{z}, z, IP) = c_{t,e}\left(z - \underline{z}\right) + c_{t,n} IP + L_t(z, IP) + \int_0^\infty R_t\left(IP - \xi_{\lambda^-}\right) \tilde{f}_{t,\lambda^-}\left(\xi_{\lambda^-}\right) d\xi_{\lambda^-}$$

It can be seen that $R_t(y)$ is convex in y; 2) $L_t(z, IP)$ is convex in z and free of IP. And similarly, we can have the following theorem.

Theorem 14.

1) $V_t(\underline{z}, z, IP)$ is convex in IP for given z, and there is a unique IP_t^* satisfying

$$\frac{\partial V_t(\underline{z}, z, IP)}{\partial IP} = c_{t,n} + \int_0^\infty R_t'\left(IP - \xi_{\lambda^-}\right) \tilde{f}_{t,\lambda^-}\left(\xi_\lambda\right) d\xi_\lambda = 0.$$

2) $V_t(\underline{z}, z, IP)$ is convex in z, and there is a unique z_t^* satisfying

$$\frac{\partial V_t(\underline{z},z,IP)}{\partial z}=c_{t,e}+h_{t,\lambda^-}-\left(p_{t,\lambda^-}+h_{t,\lambda^-}\right)\int_z^\infty \tilde{f}_{t,\lambda^-}\left(\xi_{\lambda^-}\right)d\xi_{\lambda^-}=0$$

Proof. Similar to the proof for Theorem 8.■

Suppose $z_t^* \le IP_t^*$, the optimal order rule in period $T-t+1$ is : 1) to purchase $IP_t^* - z_t^*$ units for the slow order and purchase $z_t^* - \underline{z}$ units for the fast order if $\underline{z} \le z_t^*$; 2) to purchase $IP_t^* - \underline{z}$ units for the slow order if $z_t^* < \underline{z} \le IP_t^*$ and not to purchase if $\underline{z} > IP_t^*$. Then by the definition of $V_t(\underline{z})$, we have

$$V_t(\underline{z})=\begin{cases} V_t\left(\underline{z},z_t^*,IP_t^*\right) & \text{If } \underline{z}\le z_t^* \\ V_t\left(\underline{z},\underline{z},IP_t^*\right) & \text{If } z_t^* < \underline{z} \le IP_t^* \\ V_t\left(\underline{z},\underline{z},\underline{z}\right) & \text{If } \underline{z} > IP_t^* \end{cases}.$$

Theorem 15. $V_t(\underline{z})$ is convex in $\underline{z}$ and $V_t^{'}(0)=-c_{t,e}$.

Proof. Similar to the proof of Theorem 9.■

If $IP_t^* < z_t^*$, then only the order for the fast delivery is placed and the target level IP_t^* satisfies

$$c_{t,e}+\left.\frac{\partial L_t(z,IP)}{\partial z}\right|_{z=IP}+\left.\frac{\partial L_t(z,IP)}{\partial IP}\right|_{z=IP}+\int_0^\infty \frac{\partial R_t\left(IP-\xi_{\lambda^-}\right)}{\partial IP}\tilde{f}_{t,\lambda^-}\left(\xi_{\lambda^-}\right)d\xi_{\lambda^-}=0.$$

In this case, the optimal ordering rule is: to purchase $IP_t^* - \underline{z}$ units for the fast order if $\underline{z} \le IP_t^*$ and not to purchase if $\underline{z} > IP_t^*$. It can be seen that $V_t(\underline{z})$, after making corresponding changes, is still convex in $\underline{z}$.

Following the above logic, we can show that $V_1(\underline{z})$ is convex (since $V_0(\underline{z})\equiv 0$) and $V_1^{'}(0)=-c_{1,e}$. This completes the induction.

To close the discussion for the case when shortages are backordered without penalty feature, we point out that Barnes-Schuster et al. (2002) discussed essentially a two-period variant of this case such that $\lambda = 1$, there is only one supply mode in the second period with a limit on the order quantity, and this limit is determined in the first period.

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